# AXEL-2018

## Introduction to Particle Accelerators

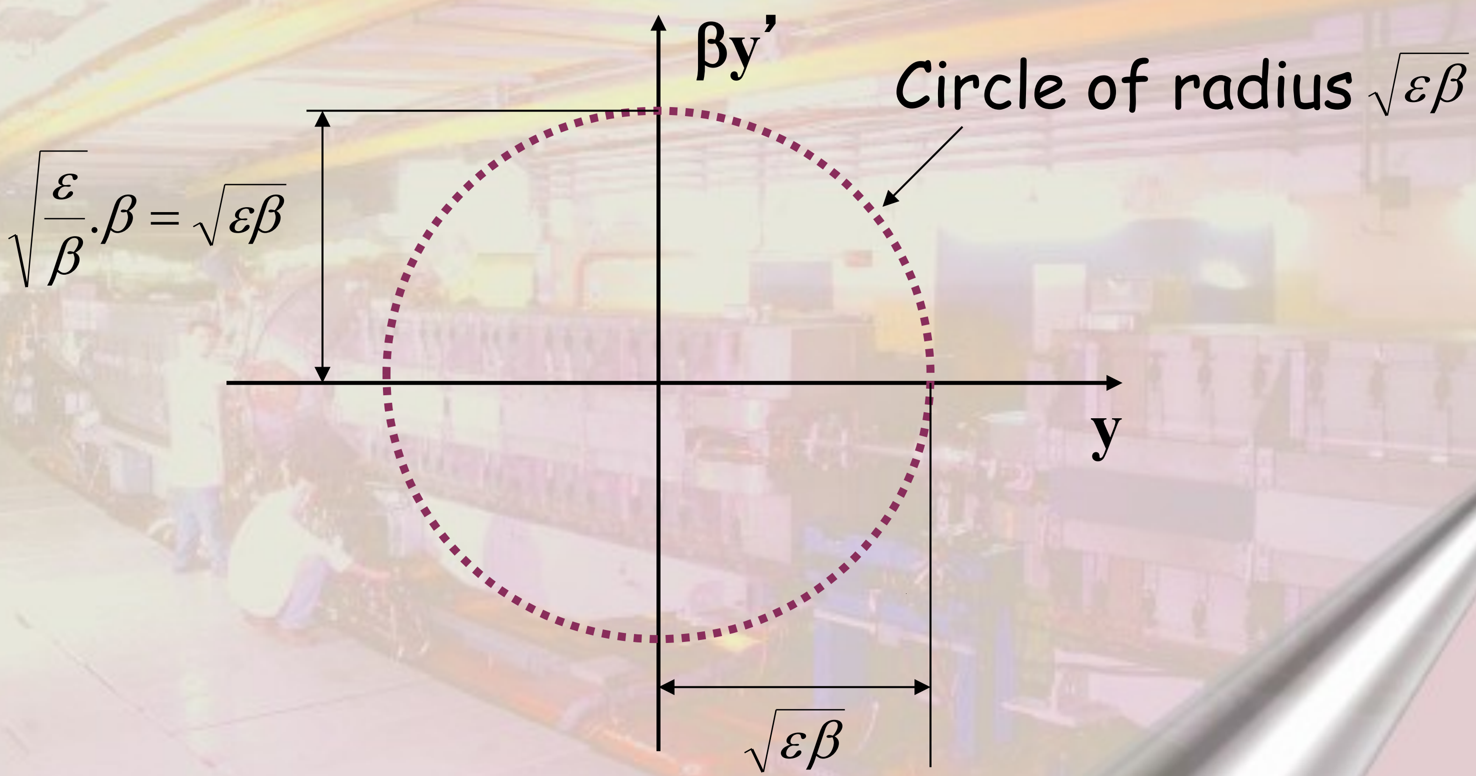
### Resonances:

- ✓ *Normalised Phase Space*
- ✓ *Dipoles, Quadrupoles, Sextupoles*
- ✓ *A more rigorous approach*
- ✓ *Coupling*
- ✓ *Tune diagram*

Rende Steerenberg (BE/OP)

7 March 2018

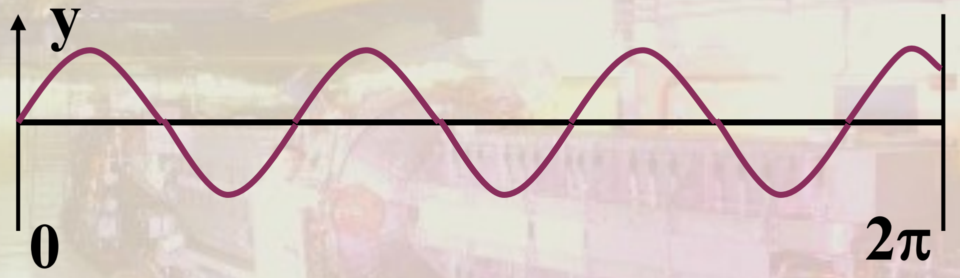
# Normalised Phase Space



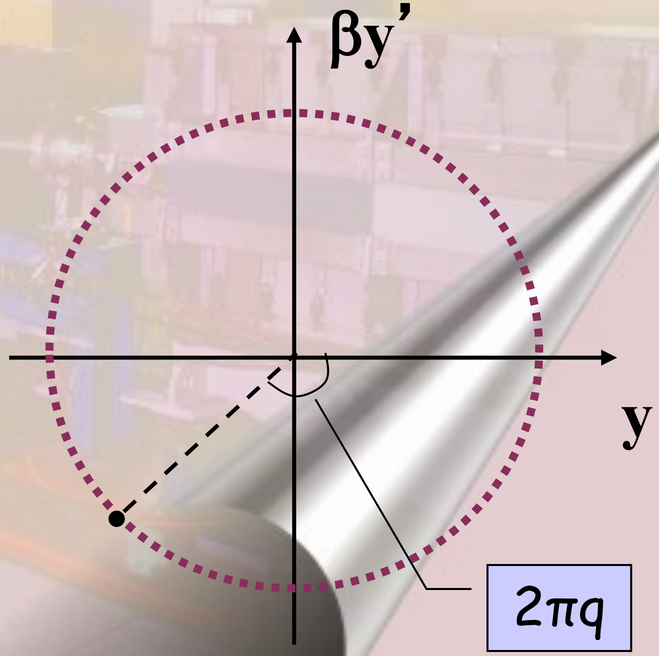
- ✓ By multiplying the y-axis by β the transverse phase space is normalised and the ellipse turns into a circle.

# Phase Space & Betatron Tune

- ✓ If we unfold a trajectory of a particle that makes one turn in our machine with a tune of Q = 3.333, we get:



- ✓ This is the same as going 3.333 time around on the circle in phase space
- ✓ The net result is 0.333 times around the circular trajectory in the normalised phase space
- ✓ q is the fractional part of Q
- ✓ So here Q= 3.333 and q = 0.333

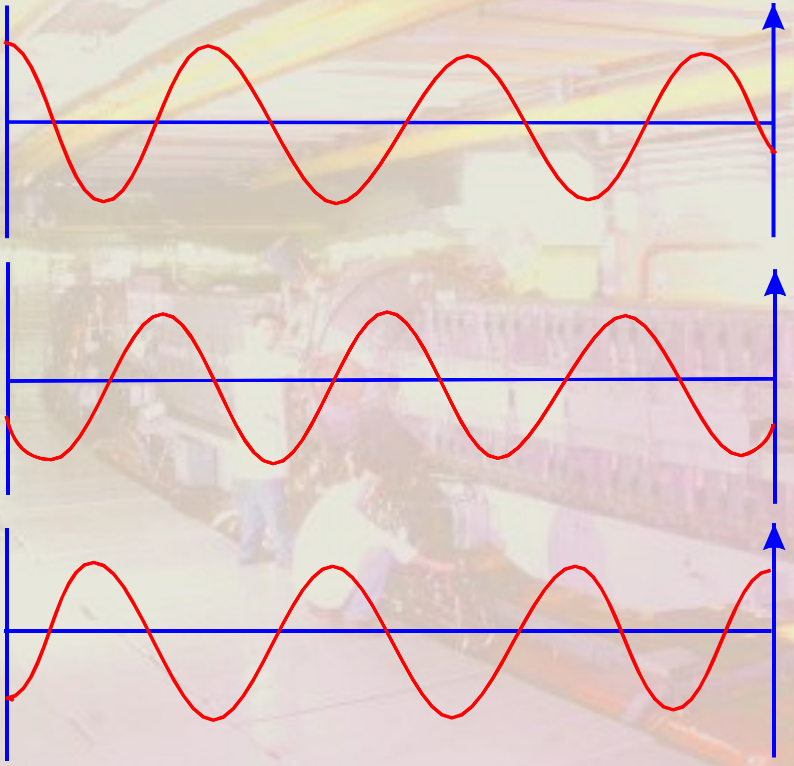


# What is a resonance?

- ✓ After a certain number of turns around the machine the phase advance of the betatron oscillation is such that the oscillation repeats itself.
- ✓ For example:
  - ✓ If the phase advance per turn is 120° then the betatron oscillation will repeat itself after 3 turns.
  - ✓ This could correspond to $Q = 3.333$ or $3Q = 10$
  - ✓ But also $Q = 2.333$ or $3Q = 7$
- ✓ The order of a resonance is defined as 'n'

$$n \times Q = \text{integer}$$

# $Q = 3.333$ in more detail



1st turn

2nd turn

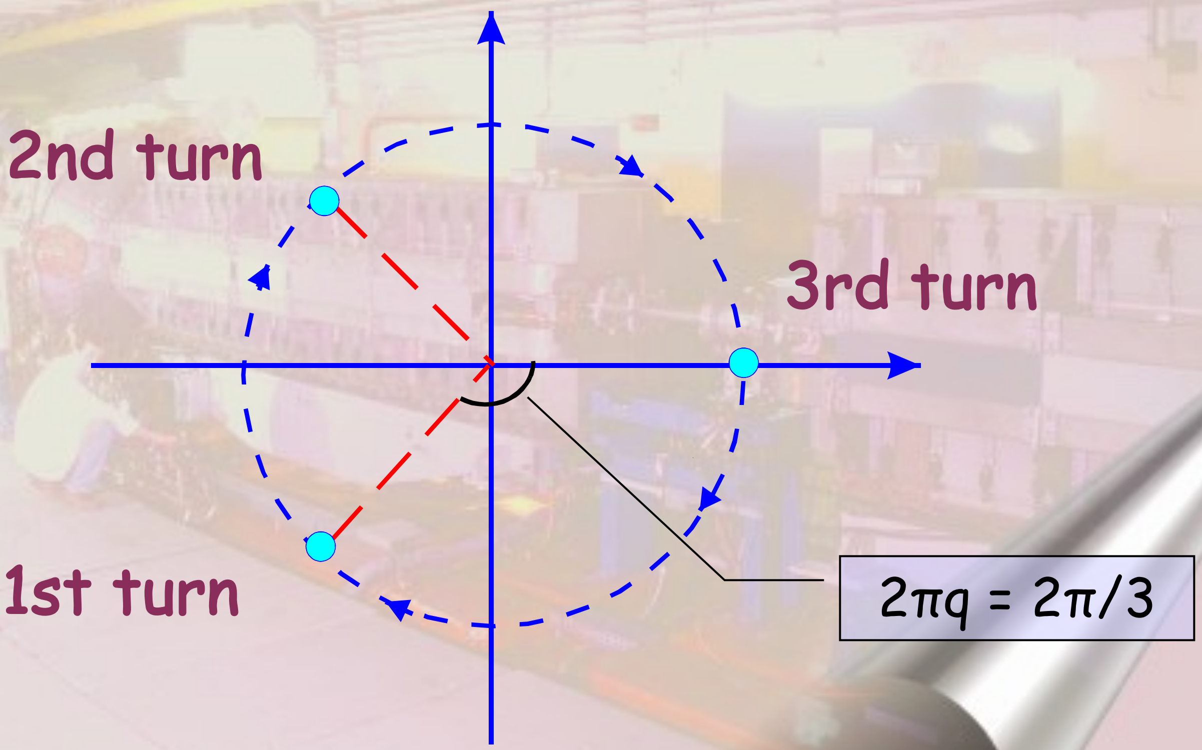
3rd turn

**Third order resonant betatron oscillation**

**$3Q = 10$, $Q = 3.333$, $q = 0.333$**

# Q = 3.333 in Phase Space

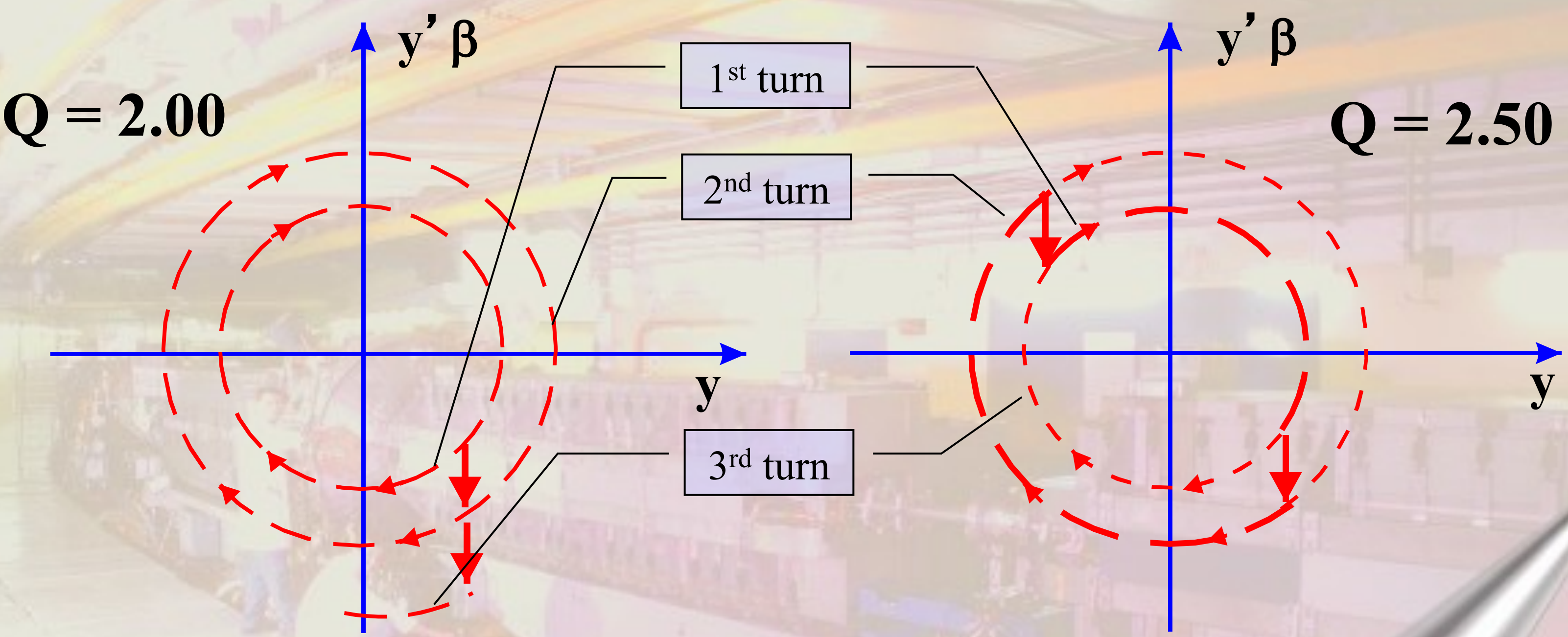
- ✓ Third order resonance on a normalised phase space plot



# Machine imperfections

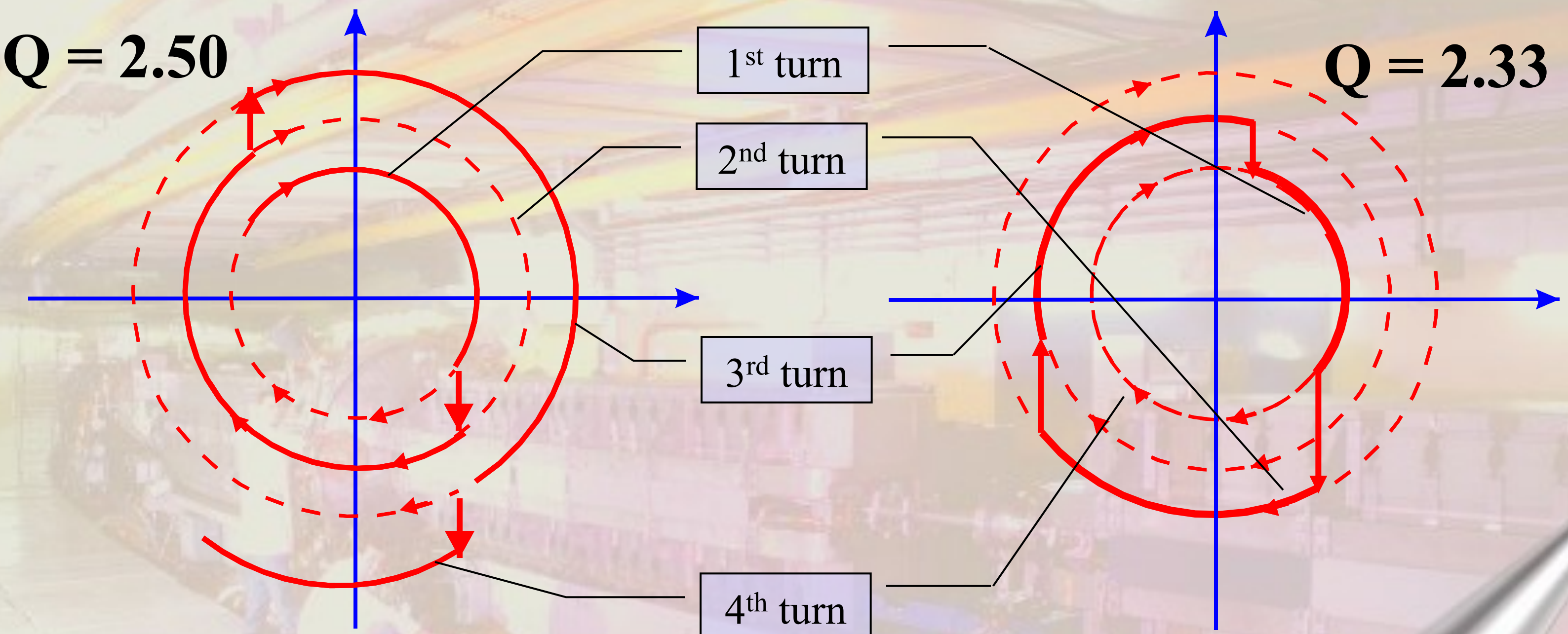
- ✓ It is not possible to construct a perfect machine.
  - ✓ Magnets can have imperfections
  - ✓ The alignment in the de machine has non zero tolerance.
  - ✓ Etc…
- ✓ So, we have to ask ourselves:
  - ✓ What will happen to the betatron oscillations due to the different field errors.
  - ✓ Therefore we need to consider errors in dipoles, quadrupoles, sextupoles, etc…
- ✓ We will have a look at the beam behaviour as a function of 'Q'
- ✓ How is it influenced by these resonant conditions?

# Dipole (deflection independent of position)



- ✓ For Q = 2.00: Oscillation induced by the dipole kick grows on each turn and the particle is lost (1st order resonance Q = 2).
- ✓ For Q = 2.50: Oscillation is cancelled out every second turn, and therefore the particle motion is stable.

# Quadrupole (deflection ∝ position)

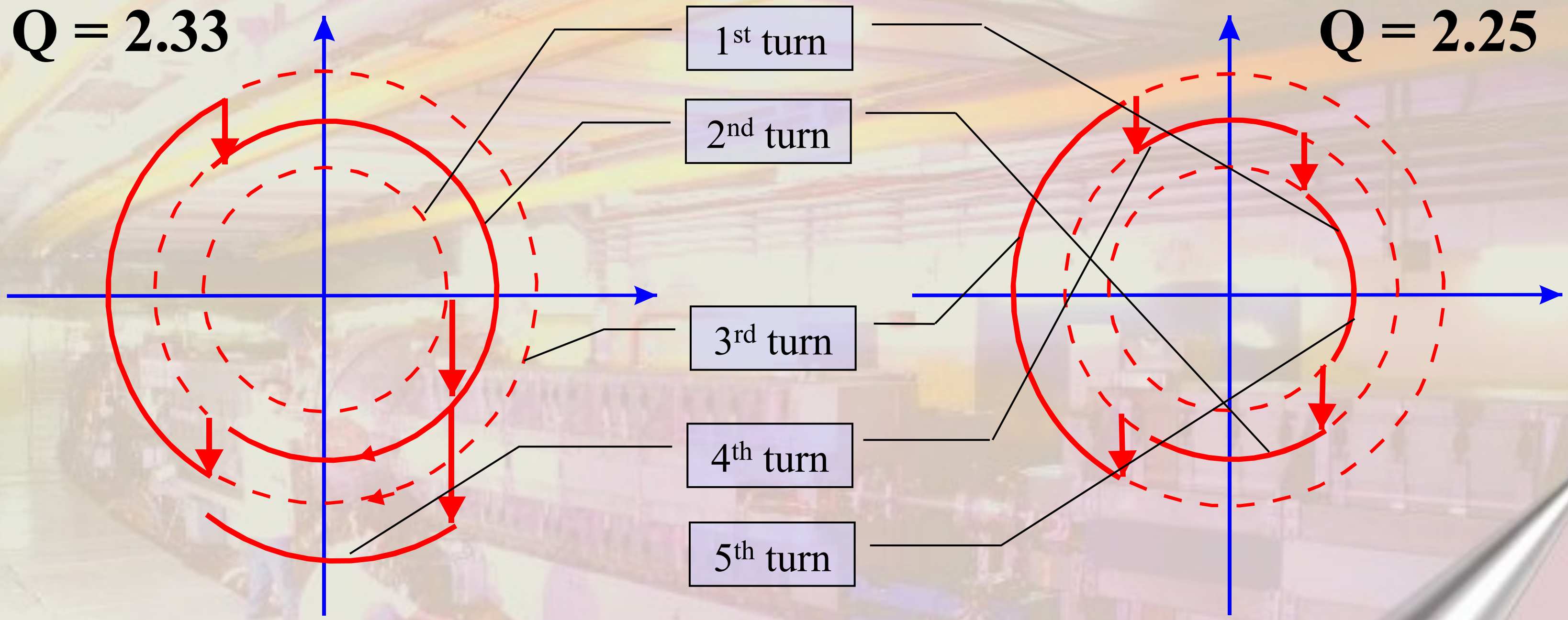


- ✓ For Q = 2.50: Oscillation induced by the quadrupole kick grows on each turn and the particle is lost

  (2nd order resonance 2Q = 5)

- ✓ For Q = 2.33: Oscillation is cancelled out every third turn, and therefore the particle motion is stable.

# Sextupole (deflection ∝ position$^2$)



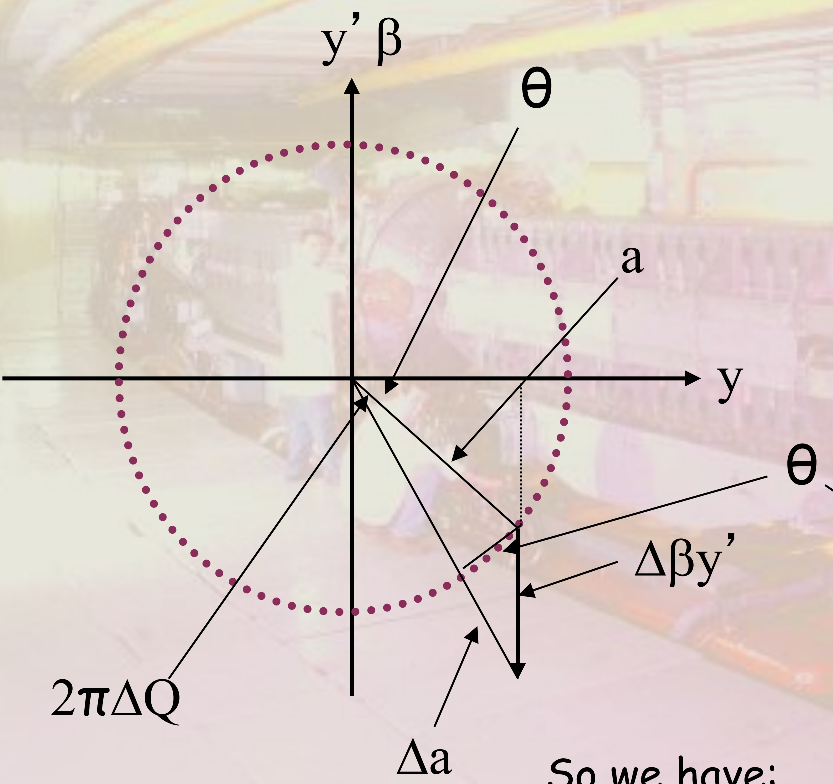
- ✓ For Q = 2.33: Oscillation induced by the sextupole kick grows on each turn and the particle is lost

  (3rd order resonance 3Q = 7)

- ✓ For Q = 2.25: Oscillation is cancelled out every fourth turn, and therefore the particle motion is stable.

# More rigorous approach (1)

- ✓ Let us try to find a **mathematical expression** for the **amplitude growth** in the case of a quadrupole error:



2πQ = phase angle over 1 turn = θ
Δβy’ = kick
a = old amplitude
**Δa = change in amplitude**
2πΔQ = change in phase

y does not change at the kick

$y = a\cos(\theta)$

In a quadrupole $\Delta y' = lky$

Only if 2πΔQ is small

So we have:

$\Delta a = \beta \Delta y' \sin(\theta) = l\beta \sin(\theta)\, a\, k \cos(\theta)$

# More rigorous approach (2)

- ✓ So we have: $\Delta a = l\cdot\beta\cdot\sin(\theta)\ a\cdot k\cdot\cos(\theta)$ $\therefore$ $\dfrac{\Delta a}{a} = \dfrac{\ell\beta k}{2}\sin(2\theta)$

Sin(θ)Cos(θ) = 1/2 Sin (2θ)

- ✓ Each turn θ advances by 2πQ
- ✓ On the nth turn θ = θ + 2nπQ
- ✓ Over many turns: $\dfrac{\Delta a}{a} = \dfrac{\ell\beta k}{2}\sum_{n=1}^{\infty}\sin\left(2\left(\theta + 2n\pi Q\right)\right)$

This term will be 'zero' as it decomposes in Sin and Cos terms and will give a series of + and – that cancel out in all cases where the fractional tune q ≠ 0.5

- ✓ So, for q = 0.5 the phase term, 2(θ + 2nπQ) is constant:

$$\sum_{n=1}^{\infty}\sin\left(2\left(\theta + 2n\pi Q\right)\right) = \infty$$

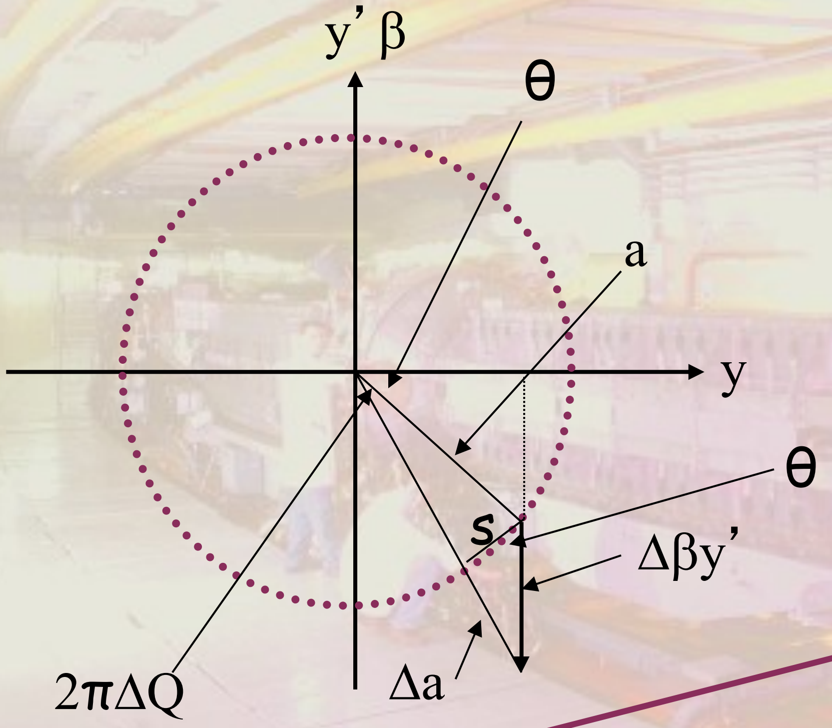
and thus: $\dfrac{\Delta a}{a} = \infty$

# More rigorous approach (3)

- ✓ In this case the amplitude will grow continuously until the particles are lost.
- ✓ Therefore we conclude as before that:
  quadrupoles excite 2nd order resonances for q=0.5
- ✓ Thus for Q = 0.5, 1.5, 2.5, 3.5,…etc……

# More rigorous approach (4)

- ✓ Let us now look at the phase θ for the same quadrupole error:



2πQ = phase angle over 1 turn = θ

Δβy' = kick

a = old amplitude

Δa = change in amplitude

**2πΔQ = change in phase**

y does not change at the kick

$$y = a\cos(\theta)$$

In a quadrupole $\Delta y' = lky$

$$\boxed{s = \Delta(\beta y')\cos\theta}$$

$$\boxed{2\pi\Delta Q = \frac{\Delta(\beta y')\cos\theta}{a}} \rightarrow \boxed{\Delta Q = \frac{1}{2\pi}\cdot\frac{\beta\cdot\cos(\theta)\cdot l\cdot a\cdot k\cdot\cos(\theta)}{a}}$$

2πΔQ << Therefore Sin(2πΔQ) ≈ 2πΔQ

# More rigorous approach (5)

- ✓ So we have: $\Delta Q = \frac{1}{2\pi} \cdot \frac{\beta \cdot \cos(\theta) \cdot l \cdot a \cdot k \cdot \cos(\theta)}{a}$
- ✓ Since: $Cos^2(\theta) = \frac{1}{2} Cos(2\theta) + \frac{1}{2}$ we can rewrite this as:

$$\Delta Q = \frac{1}{4\pi} \cdot l \cdot \beta \cdot k \cdot (\cos(2\theta) + 1)$$

, which is correct for the 1st turn

- ✓ Each turn θ advances by 2πQ
- ✓ On the nth turn θ = θ + 2nπQ
- ✓ Over many turns: $\Delta Q = \frac{1}{4\pi} \ell \beta k \left[ \underbrace{\sum_{n=1}^{\infty} \cos\left(2(\theta + 2\pi n Q)\right)}_{\text{‘zero’}} + 1 \right]$
- ✓ Averaging over many turns: $\Delta Q = \frac{1}{4\pi} \beta.k.ds$

# Stopband

- ✓ $\Delta Q=\frac{1}{4\pi}\beta.k.ds$ , which is the expression for the change in Q due to a quadrupole… (fortunately !!!)
- ✓ But note that Q changes slightly on each turn

$$\Delta Q=\frac{1}{4\pi}l\cdot\beta\cdot k(\underline{\cos(2\theta)+1})$$

Related to Q

Max variation 0 to 2

- ✓ Q has a range of values varying by: $\frac{\ell\beta k}{2\pi}$
- ✓ This width is called the **stopband** of the resonance
- ✓ So even if q is not exactly 0.5, it must not be too close, or at some point it will find itself at exactly 0.5 and 'lock on' to the resonant condition.

# Sextupole kick

- ✓ We can apply the same arguments for a sextupole:
- ✓ For a sextupole $\Delta y' = \ell k y^2$ and thus $\Delta y' = \ell k a^2 \cos^2\theta$
- ✓ We get : $\frac{\Delta a}{a} = \ell \beta k a \sin\theta \cos^2\theta = \frac{\ell\beta k a}{2}\left[\cos 3\theta + \cos\theta\right]$
- ✓ Summing over many turns gives:

$$\frac{\Delta a}{a} = \frac{\ell\beta k a}{2}\sum_{n=1}^{\infty} \underline{\cos 3(\theta + 2\pi n Q)} + \underline{\cos(\theta + 2\pi n Q)}$$

3rd order resonance term

1st order resonance term

- ✓ Sextupole excite 1st and 3rd order resonance

q = 0

q = 0.33

# Octupole kick

- ✓ We can apply the same arguments for an octupole:
- ✓ For an octupole $\Delta y' = \ell k y^3$ and thus $\Delta y' = \ell k a^3 \cos^3 \theta$
- ✓ We get : $\dfrac{\Delta a}{a} = \ell \beta k a^2 \sin\theta \cos^3 \theta$
- ✓ Summing over many turns gives:

$$\frac{\Delta a}{a} \propto a^2(\cos 4(\theta+2\pi nQ) + \cos 2(\theta+2\pi nQ))$$

4th order resonance term

2nd order resonance term

Amplitude squared

q = 0.5

q = 0.25

- ✓ Octupolar errors excite 2nd and 4th order resonance and are very important for larger amplitude particles.

Can restrict dynamic aperture

# Resonance summary

- ✓ **Quadrupoles** excite 2nd order resonances
- ✓ **Sextupoles** excite 1st and 3rd order resonances
- ✓ **Octupoles** excite 2nd and 4th order resonances

- ✓ This is true for small amplitude particles and low strength excitations

- ✓ However, for stronger excitations sextupoles will excite higher order resonance's (non-linear)

# Coupling

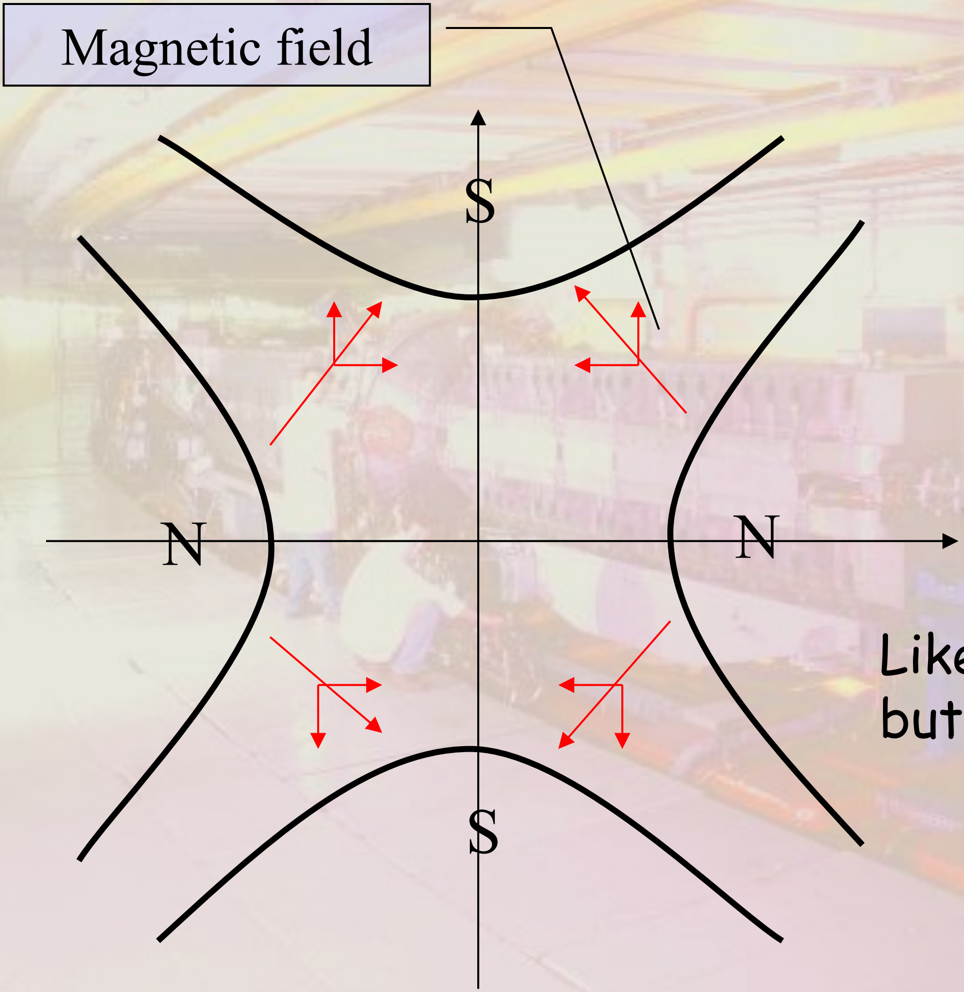
- ✓ Coupling is a phenomena, which converts betatron motion from one plane (horizontal or vertical) into motion in the other plane.

- ✓ Fields that will excite coupling are:

  - ✓ Skew quadrupoles, which are normal quadrupoles, but tilted by 45° about it's longitudinal axis.

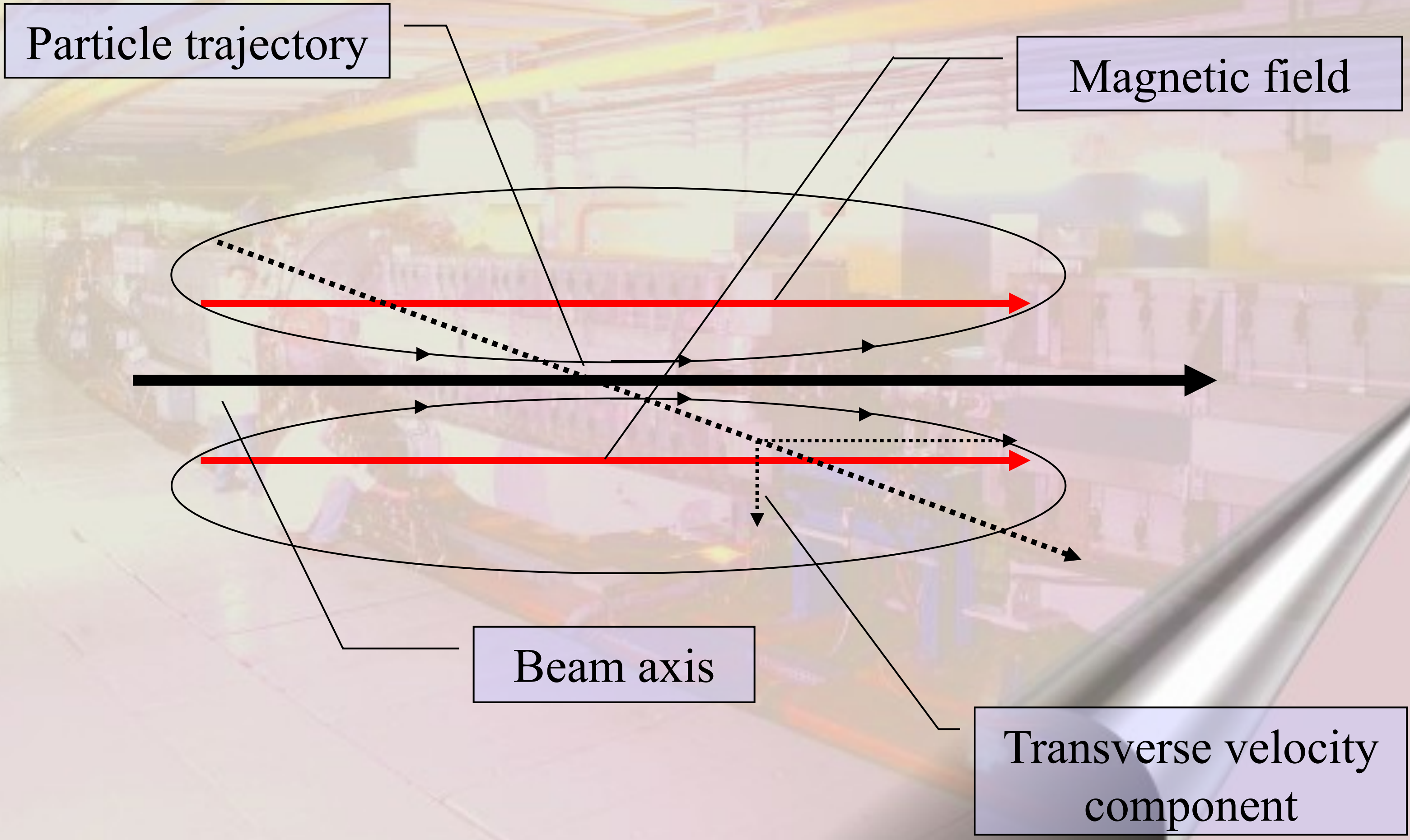
  - ✓ Solenoidal (longitudinal magnetic field)

# Skew Quadrupole



Like a normal quadrupole, but then tilted by 45°

# Solenoid; longitudinal field (2)



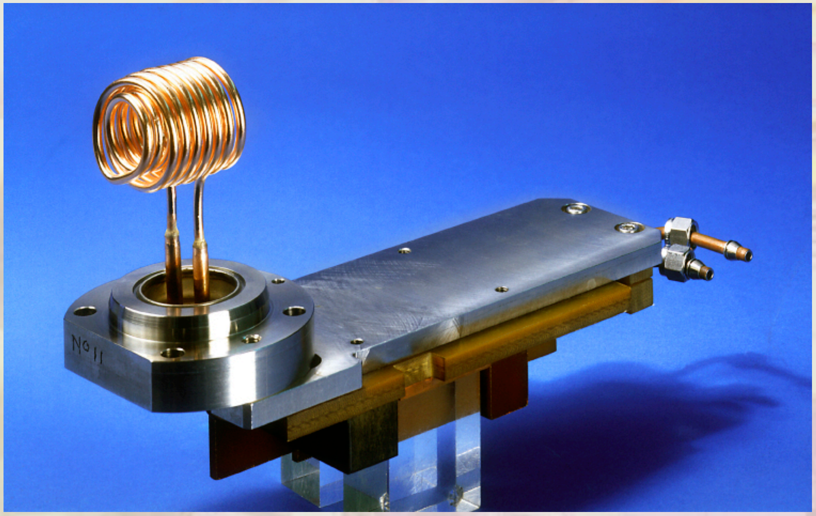
Particle trajectory

Magnetic field

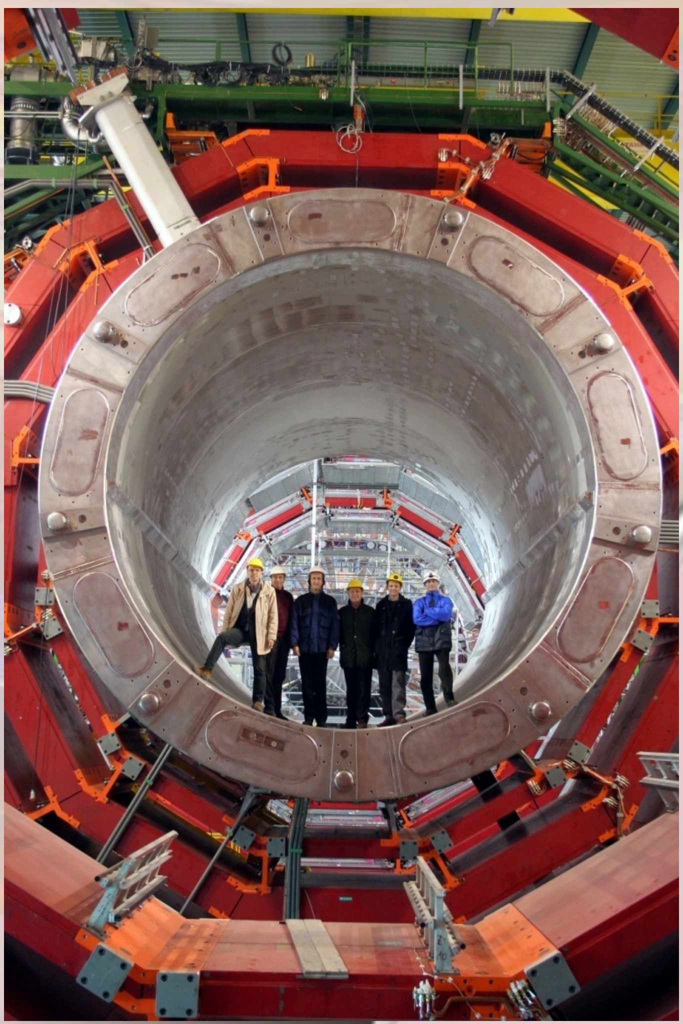
Beam axis

Transverse velocity component

# Solenoid; longitudinal field (2)



Above:
The LPI solenoid that was used for the initial focusing of the positrons.
It was pulsed with a current of 6 kA for some 7 us, it produced a longitudinal magnetic field of 1.5 T.



At the right:
The somewhat bigger CMS solenoid

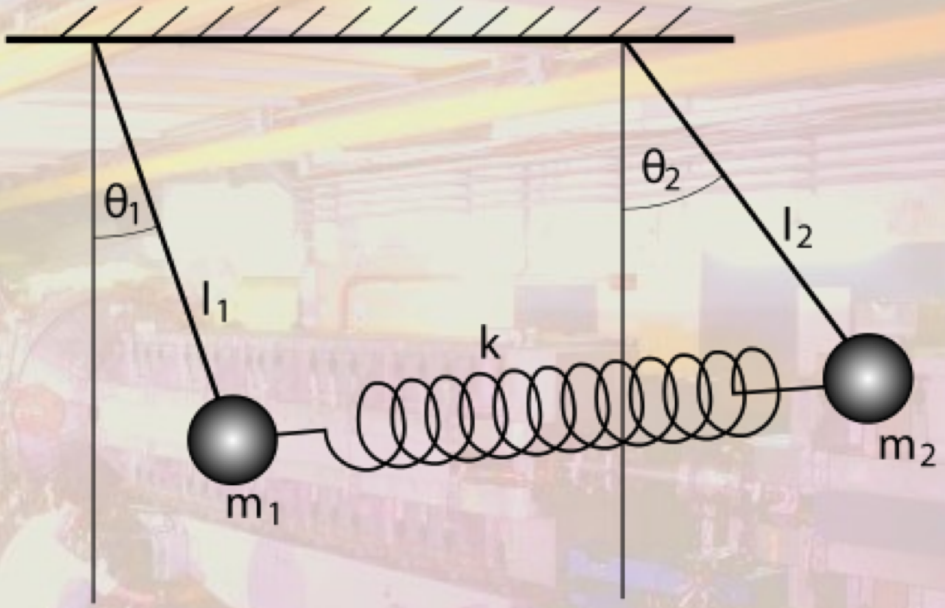
# Coupling and Resonance

- ✓ This coupling means that one can transfer oscillation energy from one transverse plane to the other.
- ✓ Exactly as for linear resonances there are resonant conditions.

$$nQ_h \pm mQ_v = \text{integer}$$

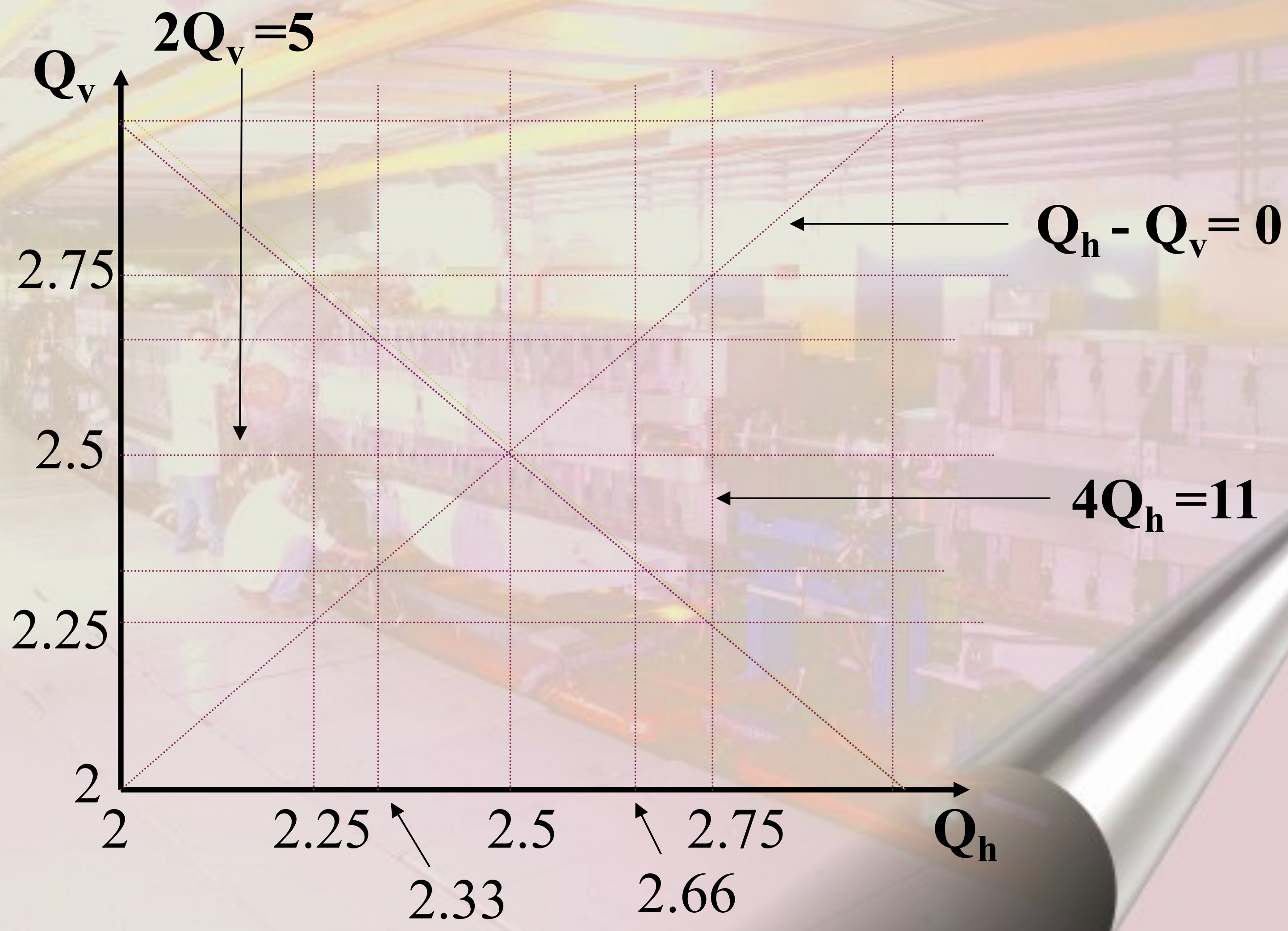
- ✓ If we meet one of these conditions the transverse oscillation amplitude will again grow in an uncontrolled way.

# A mechanical equivalent

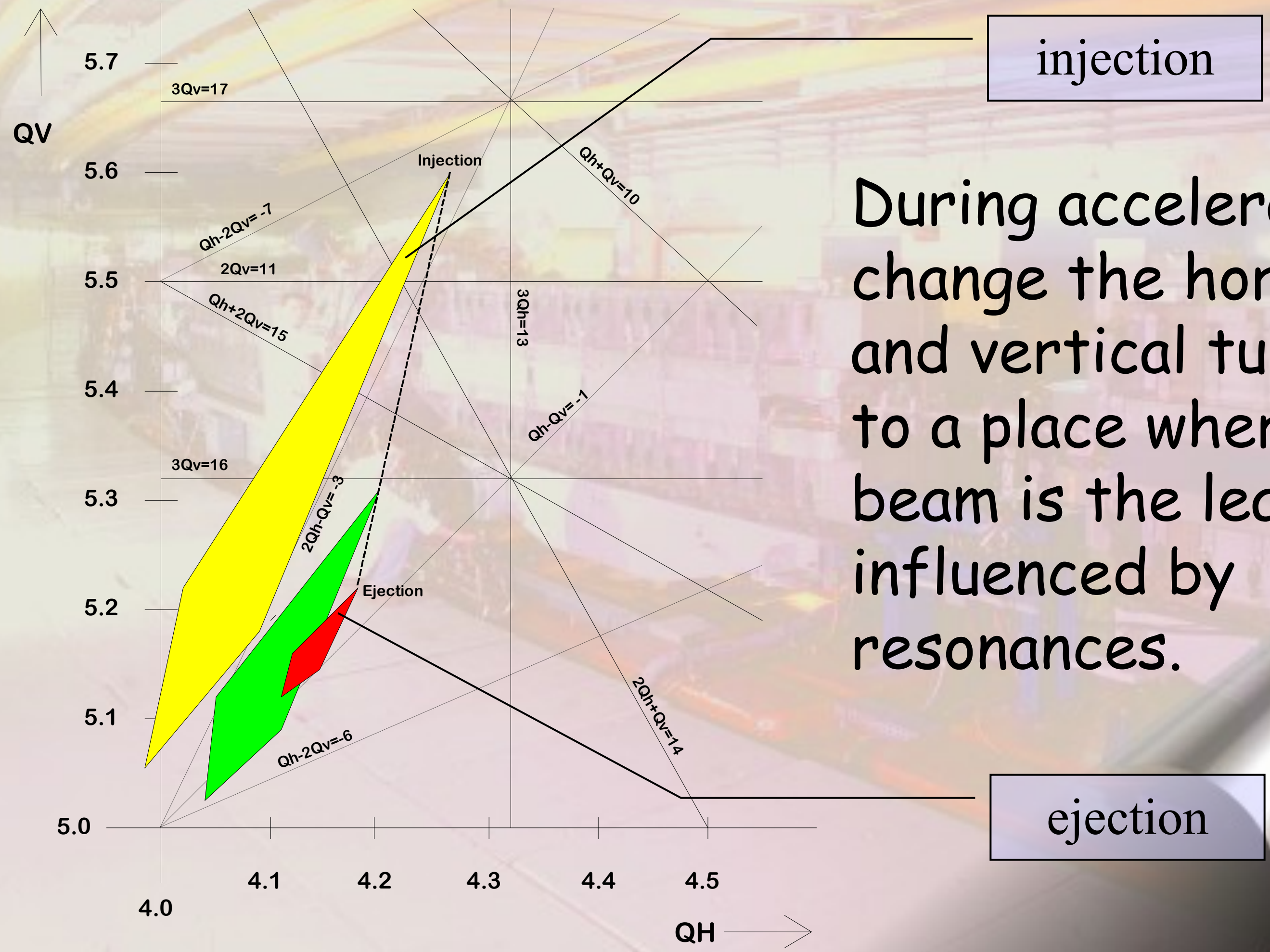


- We can transfer oscillation energy from one pendulum to the other depending on the strength 'k' of the spring

# General tune diagram

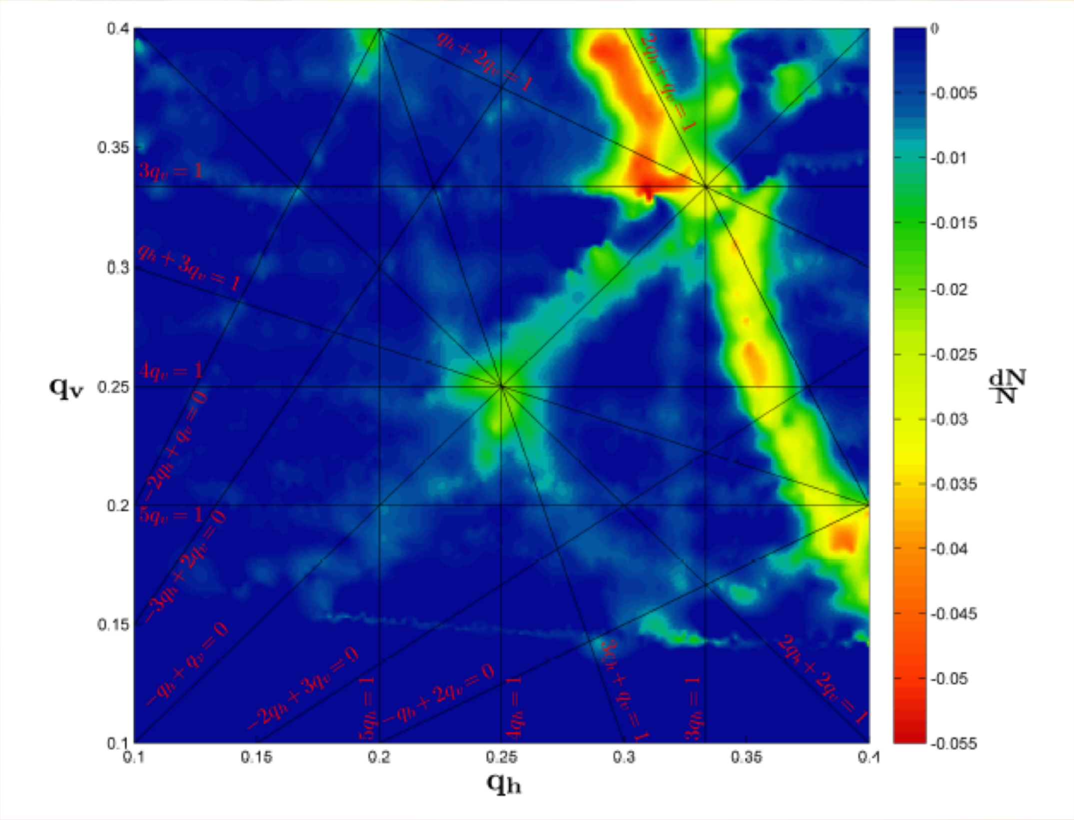


# Realistic tune diagram



During acceleration we change the horizontal and vertical tune to a place where the beam is the least influenced by resonances.

# Measured tune diagram



Move a large emittance beam around in this tune diagram and measure the beam losses.

Not all resonance lines are harmful.

# Conclusion

- ✓ There are many things in our machine, which will excite resonances:
  - ✓ The magnets themselves
  - ✓ Unwanted higher order field components in our magnets
  - ✓ Tilted magnets
  - ✓ Experimental solenoids (LHC experiments)
- ✓ The trick is to reduce and compensate these effects as much as possible and then find some point in the tune diagram where the beam is stable.

# Questions….,Remarks…?