AXEL-2018
Introduction to Particle Accelerators

Lattice calculations:

- ✓ *Lattices*
- ✓ *Tune Calculations*
- ✓ *Dispersion*
- ✓ *Momentum Compaction*
- ✓ *Chromaticity*
- ✓ *Sextupoles*

Rende Steerenberg (BE/OP)

6 March 2018

A quick recap……

- ✓ We solved Hill's equation, which led us to the definition of transverse emittance and allowed us to describe particle motion in transverse phase space in terms of β, α, etc…
- ✓ We constructed the Transport Matrices corresponding to drift spaces and quadrupoles.
- ✓ **Now we must combine these matrices with the solution of Hill's equation to evaluate β, α, etc…**

Matrices & Hill's equation

- We can multiply the matrices of our drift spaces and quadrupoles together to form a transport matrix that describes a larger section of our accelerator.
- These matrices will move our particle from one point $(x(s_1),x'(s_1))$ on our phase space plot to another $(x(s_2),x'(s_2))$, as shown in the matrix equation below.

$$\begin{pmatrix} x(s_2) \\ x'(s_2) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix}$$

- The elements of this matrix are fixed by the elements through which the particles pass from point s_1 to point s_2.
- However, we can also express (x, x') as solutions of Hill's equation.

$$x = \sqrt{\varepsilon.\beta}\cos\phi \quad \text{and} \quad x' = -\alpha\sqrt{\varepsilon/\beta}\cos\phi - \sqrt{\varepsilon/\beta}\sin\phi$$

Matrices & Hill's equation (2)

$$x = \sqrt{\varepsilon.\beta}\cos(\mu+\phi)$$

$$x = \sqrt{\varepsilon.\beta}\cos\phi$$

$$\begin{pmatrix} x(s_2) \\ x'(s_2) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix}$$

$$x' = -\alpha\sqrt{\varepsilon/\beta}\cos(\mu+\phi) - \sqrt{\varepsilon/\beta}\sin(\mu+\phi)$$

$$x' = -\alpha\sqrt{\varepsilon/\beta}\cos\phi - \sqrt{\varepsilon/\beta}\sin\phi$$

- ✓ Assume that our transport matrix describes a complete turn around the machine.
- ✓ Therefore : $\beta(s_2) = \beta(s_1)$
- ✓ Let μ be the change in betatron phase over one complete turn.
- ✓ Then we get for $x(s_2)$:

$$x(s_2) = \sqrt{\varepsilon.\beta}\cos(\mu+\phi) = a\sqrt{\varepsilon.\beta}\cos\phi - b\alpha\sqrt{\varepsilon/\beta}\cos\phi - b\sqrt{\varepsilon/\beta}\sin\phi$$

Matrices & Hill's equation (3)

- So, for the position x at s2 we have…

$$\sqrt{\varepsilon.\beta}\,\underbrace{\cos(\mu+\phi)}_{\cos\phi\cos\mu-\sin\phi\sin\mu}=a\sqrt{\varepsilon.\beta}\cos\phi-b\alpha\sqrt{\varepsilon/\beta}\cos\phi-b\sqrt{\varepsilon/\beta}\sin\phi$$

- Equating the 'sin' terms gives: $-\sqrt{\varepsilon.\beta}\sin\mu\sin\phi=-b\sqrt{\varepsilon/\beta}\sin\phi$
- Which leads to: $b=\beta\sin\mu$
- Equating the 'cos' terms gives: $$\sqrt{\varepsilon.\beta}\cos\mu\cos\phi=a\sqrt{\varepsilon.\beta}\cos\phi-\alpha\sqrt{\varepsilon.\beta}\sin\mu\cos\phi$$
- Which leads to: $a=\cos u+\alpha\sin\mu$
- We can repeat this for **c** and **d**.

Matrices & Twiss parameters

- ✓ Remember previously we defined:

$$\alpha = \frac{-\beta'}{2} = -\omega\omega'$$

$$\beta = \omega^2$$

$$\gamma = \frac{1+\alpha^2}{\beta}$$

- ✓ These are called **TWISS parameters**

- ✓ Remember also that μ is the total betatron phase advance over one complete turn is.

$$Q = \frac{\mu}{2\pi}$$

Number of betatron oscillations per turn

- ✓ Our transport matrix becomes now:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

Lattice parameters

$$\begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

- ✓ This matrix describes one complete turn around our machine and will vary depending on the starting point (s).
- ✓ If we start at any point and multiply all of the matrices representing each element all around the machine we can calculate α, β, γ and μ for that specific point, which then will give us $\beta(s)$ and Q
- ✓ If we repeat this many times for many different initial positions (s) we can calculate our Lattice Parameters for all points around the machine.

Lattice calculations and codes

- Obviously μ (or Q) is not dependent on the initial position 's', but we can calculate the change in betatron phase, dμ, from one element to the next.
- Computer codes like "MAD" or "Transport" vary lengths, positions and strengths of the individual elements to obtain the desired beam dimensions or envelope '$\beta(s)$' and the desired 'Q'.
- Often a machine is made of many individual and identical sections (FODO cells). In that case we only calculate a single cell and not the whole machine, as the the functions $\beta(s)$ and dμ will repeat themselves for each identical section.
- The insertion sections have to be calculated separately.

The β(s) and Q relation.

- ✓ $Q = \frac{\mu}{2\pi}$,where μ = ΔΦ over a complete turn

- ✓ But we also found: $\frac{d\phi(s)}{ds} = \frac{1}{\beta(s)}$

Over one complete turn

- ✓ This leads to: $Q = \frac{1}{2\pi}\int_{o}^{s}\frac{ds}{\beta(s)}$

- ✓ Increasing the focusing strength decreases the size of the beam envelope (β) and increases Q and vice versa.

Tune corrections

- ✓ What happens if we change the focusing strength slightly?
- ✓ The Twiss matrix for our 'FODO' cell is given by:

$$\begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

- ✓ Add a small QF quadrupole, with strength dK and length ds.
- ✓ This will modify the 'FODO' lattice, and add a horizontal focusing term:

$$\begin{pmatrix} 1 & 0 \\ -dkds & 1 \end{pmatrix} \qquad dk = \frac{dK}{(B\rho)} \qquad f = \frac{(B\rho)}{dKds}$$

- ✓ The new Twiss matrix representing the modified lattice is:

$$\begin{pmatrix} 1 & 0 \\ -dkds & 1 \end{pmatrix}\begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

Tune corrections (2)

- ✓ This gives $\begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -dkds(\cos\mu + \sin\mu) - \gamma\sin\mu & -dkds\beta\sin\mu + \cos\mu - \alpha\sin\mu \end{pmatrix}$

- ✓ This extra quadrupole will modify the phase advance μ for the FODO cell.

New phase advance

$$\mu_1 = \mu + \mathrm{d}\mu$$

Change in phase advance

- ✓ If dμ is small then we can ignore changes in β

- ✓ So the new Twiss matrix is just:

$$\begin{pmatrix} \cos\mu_1 + \alpha\sin\mu_1 & \beta\sin\mu_1 \\ -\gamma\sin\mu_1 & \cos\mu_1 - \alpha\sin\mu_1 \end{pmatrix}$$

Tune corrections (3)

- These two matrices represent the same FODO cell therefore:

$$\begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -dkds(\cos\mu + \sin\mu) - \gamma\sin\mu & -dkds\,\beta\sin\mu + \cos\mu - \alpha\sin\mu \end{pmatrix}$$

- Which equals:

$$\begin{pmatrix} \cos\mu_1 + \alpha\sin\mu_1 & \beta\sin\mu_1 \\ -\gamma\sin\mu_1 & \cos\mu_1 - \alpha\sin\mu_1 \end{pmatrix}$$

- Combining and compare the first and the fourth terms of these two matrices gives:

$$2\cos\mu_1 = 2\cos\mu - dk\ ds\,\beta\sin\mu$$

Only valid for change in b <<

Tune corrections (4)

$$2\cos\mu_1 = 2\cos\mu - dk\ ds\,\beta\sin\mu$$

Remember $\mu_1 = \mu + d\mu$ and $d\mu$ is small

$$2\cos\mu - 2\sin\mu\, d\mu$$

$$2\sin\mu\, d\mu = dk\, ds\, \beta \sin\mu$$

$$d\mu = \frac{1}{2} dk\, ds\, \beta$$

,but: $dQ = d\mu/2\pi$

In the **horizontal plane** this is a **QF**

$$dQh = +\frac{1}{4\pi} dk.ds.\beta h$$

If we follow the same reasoning for both transverse planes for both **QF** and **QD** quadrupoles

$$dQv = +\frac{1}{4\pi}\beta v.dk_D.ds_D - \frac{1}{4\pi}\beta v.dk_F.ds_F$$

$$dQh = -\frac{1}{4\pi}\beta h.dk_D.ds_D + \frac{1}{4\pi}\beta h.dk_F.ds_F$$

QD **QF**

Tune corrections (5)

Let $\mathbf{dk_F} = \mathbf{dk}$ for **QF** and $\mathbf{dk_D} = \mathbf{dk}$ for **QD**

$\beta_{hF}, \beta_{vF} = \beta$ at **QF** and $\beta_{hD}, \beta_{vD} = \beta$ at **QD**

Then:

$$\begin{pmatrix} dQv \\ dQh \end{pmatrix} = \begin{pmatrix} \frac{1}{4\pi}\beta_{vD} & \frac{-1}{4\pi}\beta_{vF} \\ \frac{-1}{4\pi}\beta_{hD} & \frac{1}{4\pi}\beta_{hF} \end{pmatrix} \begin{pmatrix} dk_D ds \\ dk_F ds \end{pmatrix}$$

This matrix relates the change in the tune to the change in strength of the quadrupoles.
We can invert this matrix to calculate change in quadrupole field needed for a given change in tune

Dispersion (1)

- ✓ Until now we have assumed that our beam has no energy or momentum spread:

$$\frac{\Delta E}{E} = 0 \quad \text{and} \quad \frac{\Delta p}{p} = 0$$

- ✓ Different energy or momentum particles have different radii of curvature (ρ) in the main dipoles.
- ✓ These particles no longer pass through the quadrupoles at the same radial position.
- ✓ Quadrupoles act as dipoles for different momentum particles.
- ✓ Closed orbits for different momentum particles are different.
- ✓ This horizontal displacement is expressed as the dispersion function D(s)
- ✓ D(s) is a function of 's' exactly as β(s) is a function of 's'

Dispersion (2)

- ✓ The displacement due to the change in momentum at any position (s) is given by:

$$\Delta x(s) = D(s).\frac{\Delta p}{p}$$

Local radial displacement due to momentum spread

Dispersion function

- ✓ D(s) the dispersion function, is calculated from the lattice, and has the unit of meters.
- ✓ The beam will have a finite horizontal size due to it's momentum spread.
- ✓ In the majority of the cases we have no vertical dipoles, and so D(s)=0 in the vertical plane.

Momentum compaction factor

- The change in orbit with the changing momentum means that the average length of the orbit will also depend on the beam momentum.
- This is expressed as the momentum compaction factor, α_p, where:

$$\frac{\Delta r}{r} = \alpha_p \frac{\Delta p}{p}$$

- α_p tells us about the change in the length of radius of the closed orbit for a change in momentum.

Chromaticity

- The focusing strength of our quadrupoles depends on the beam momentum, 'p'

$$k = \frac{dBy}{dx} \times \frac{1}{B\rho} \leftarrow 3.3356\,p$$

- Therefore a spread in momentum causes a spread in focusing strength

$$\frac{\Delta k}{k} = -\frac{\Delta p}{p}$$

- But Q depends on the 'k' of the quadrupoles

$$\frac{\Delta Q}{Q} \alpha \frac{\Delta p}{p} \longrightarrow \frac{\Delta Q}{Q} = \xi \frac{\Delta p}{p}$$

- The constant here is called : Chromaticity

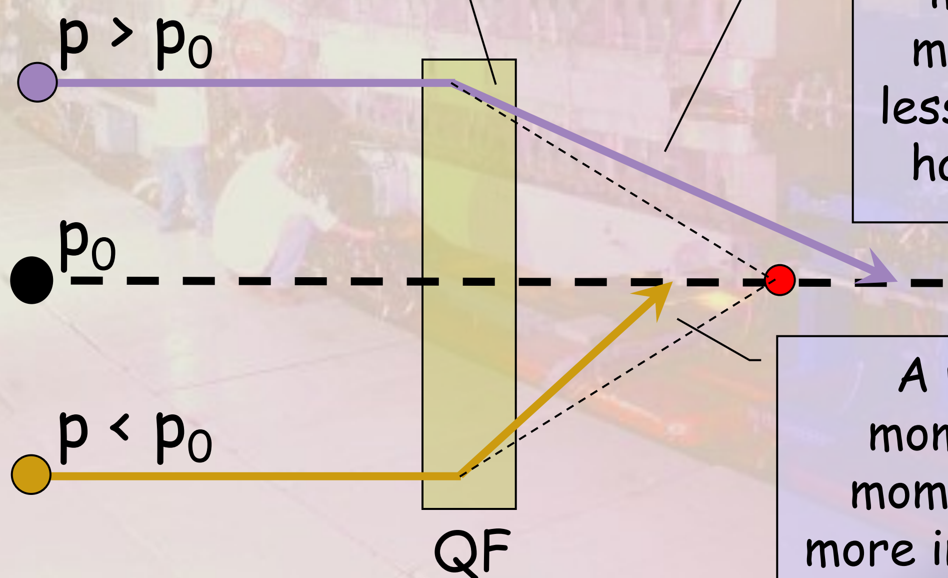
Chromaticity visualized

- ✓ The chromaticity relates the tune spread of the transverse motion with the momentum spread in the beam.

Focusing quadrupole in horizontal plane

$$\frac{\Delta Q}{Q} = \xi \frac{\Delta p}{p}$$

A particle with a higher momentum as the central momentum will be deviated less in the quadrupole and will have a lower betatron tune



A particle with a lower momentum as the central momentum will be deviated more in the quadrupole and will have a higher betatron tune

Chromaticity calculated

- ✓ Remember $\Delta Q = \frac{1}{4\pi}\left(\beta dk ds\right)$ and $\frac{\Delta k}{k} = -\frac{\Delta p}{p}$ $\Rightarrow$ $\Delta k = -k\frac{\Delta p}{p}$

The gradient seen by the particle depends on its momentum

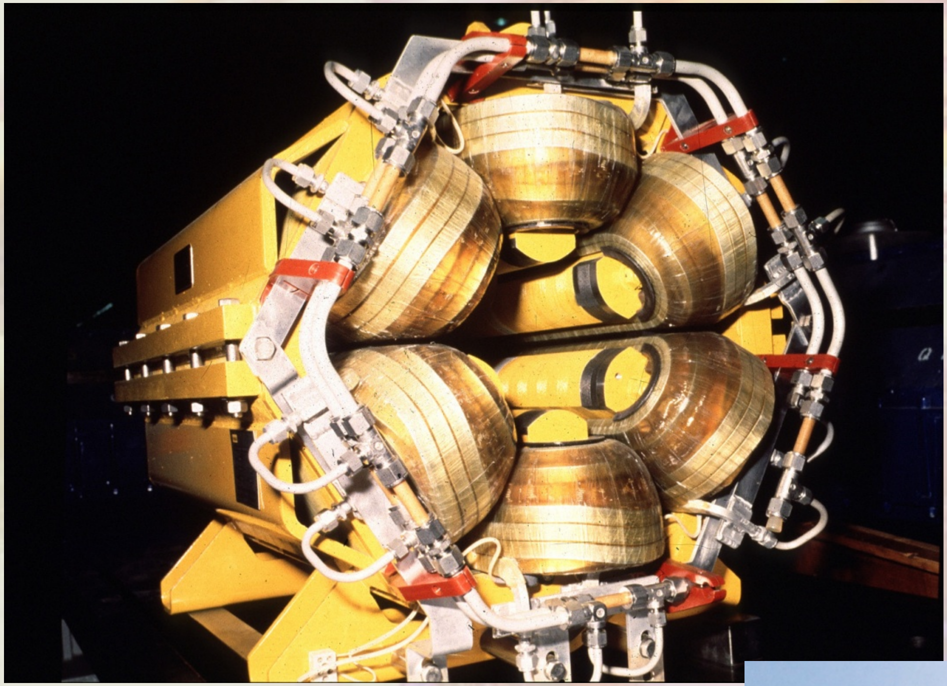
- ✓ Therefore $\frac{\Delta Q}{Q} = -\frac{1}{4\pi}\left(\beta\frac{k}{Q}ds\right)\frac{\Delta p}{p}$

- ✓ This term is the Chromaticity ξ

- ✓ To correct this tune spread we need to increase the quadrupole focusing strength for higher momentum particles, and decrease it for lower momentum particles.

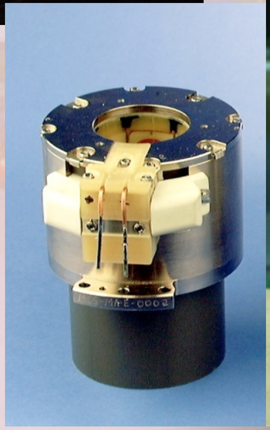
- ✓ This we will obtain using a **Sextupole** magnet

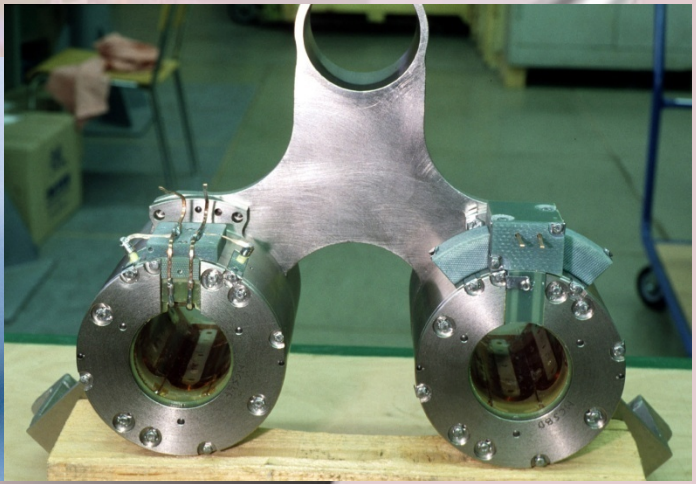
Sextupole Magnets



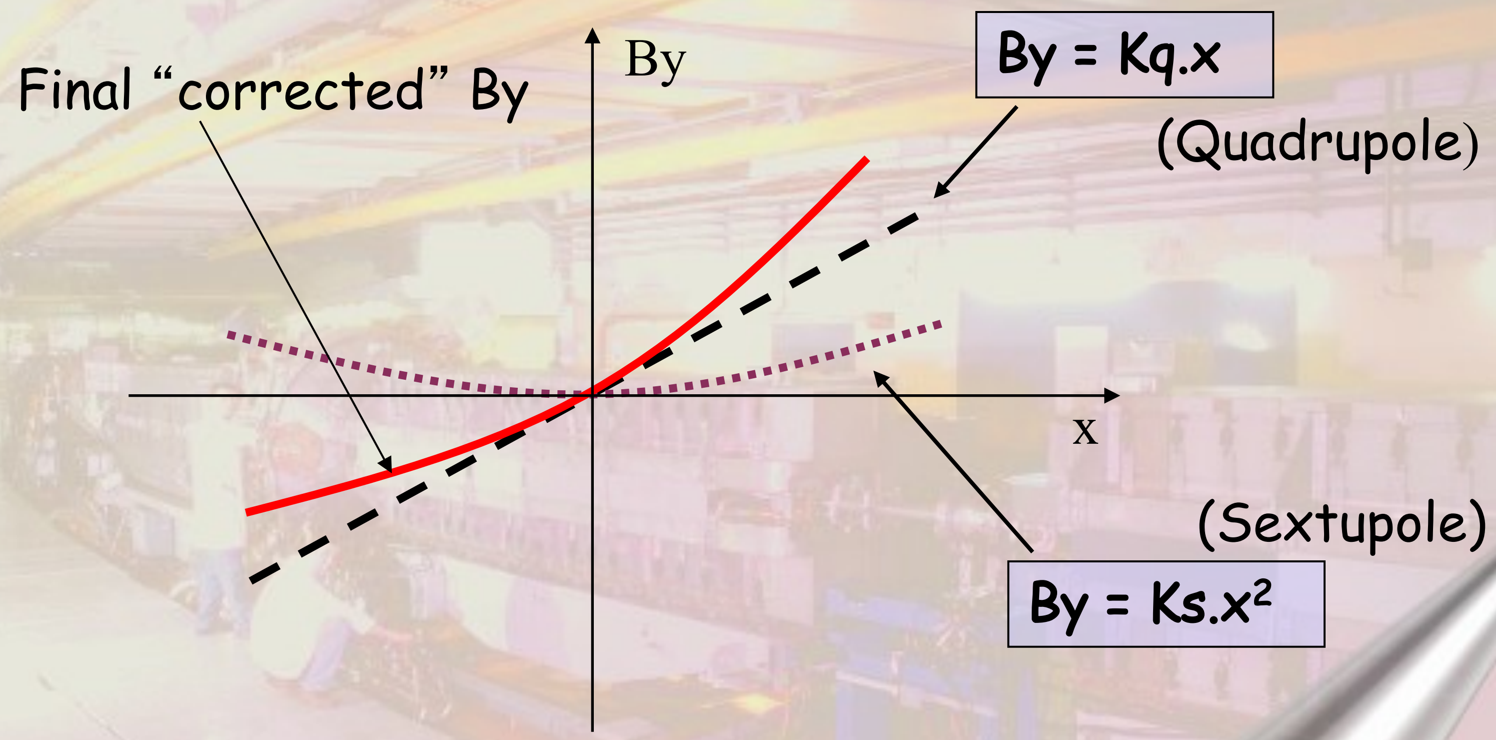
- ✓ Conventional Sextupole from LEP, but looks similar for other 'warm' machines.
- ✓ ~ 1 meter long and a few hundreds of kg.

- ✓ Correction Sextupole of the LHC
- ✓ 11cm, 10 kg, 500A at 2K for a field of 1630 T/m^2





Chromaticity correction



- ✓ Vertical magnetic field versus horizontal displacement in a quadrupole and a sextupole.

Chromaticity correction (2)

- ✓ The effect of the sextupole field is to increase the magnetic field of the quadrupoles for the positive ‘x’ particles and decrease the field for the negative ‘x’ particles.
- ✓ However, the dispersion function, D(s), describes how the radial position of the particles change with momentum.
- ✓ Therefore the sextupoles will alter the focusing field seen by the particles as a function of their momentum.
- ✓ This we can use to compensate the natural chromaticity of the machine.

Sextupole & Chromaticity

- In a sextupole for y = 0 we have a field By = C.x²
- Now calculate 'k' the focusing gradient as we did for a quadrupole:

$$k = \frac{1}{(B\rho)} \frac{dB_y}{dx}$$

- Using $B_y = Cx^2$ which after differentiating gives $\frac{dB_y}{dx} = 2Cx$
- For k we now write $k = \frac{1}{(B\rho)} 2Cx$
- We conclude that 'k' is no longer constant, as it depends on 'x'
- So for a Δx we get $\Delta k = \frac{2C}{(B\rho)} \Delta x$ and we know that $\Delta x = D(s)\frac{\Delta p}{p}$
- Therefore

$$\Delta k = 2C \times \frac{D(s)}{(B\rho)} \times \frac{\Delta p}{p}$$

Sextupole & Chromaticity

- We know that the tune changes with : $\Delta Q = \frac{1}{4\pi}\beta(s)dkds$
- Where: $ds = \text{sextupole length}$ and $dk = \Delta k = 2C \times \frac{D(s)}{(B\rho)} \times \frac{\Delta p}{p}$
- Remember $B = C \cdot x^2$ with $C = \frac{1}{2}\frac{d^2By}{dx^2}$
- The effect of a sextupole with length l on the particle tune Q as a function of Δp/p is given by:

$$\frac{\Delta Q}{Q} = \frac{1}{4\pi}\underline{\ell\beta(s)\frac{d^2By}{dx^2}\frac{D(s)}{(B\rho)Q}}\frac{\Delta p}{p}$$

- If we can make this term exactly balance the natural chromaticity then we will have solved our problem.

Sextupole & Chromaticity (2)

- There are two chromaticities:
  - horizontal → ξ_h
  - vertical → ξ_v
- However, the effect of a sextupole depends on $\beta(s)$, which varies around the machine
- Two types of sextupoles are used to correct the chromaticity.
  - One (SF) is placed near QF quadrupoles where β_h is large and β_v is small, this will have a large effect on ξ_h
  - Another (SD) placed near QD quadrupoles, where β_v is large and β_h is small, will correct ξ_v
- Also sextupoles should be placed where D(s) is large, in order to increase their effect, since Δk is proportional to D(s)

Questions….,Remarks…?

Hill's equation

Lattices and tune corrections

Sextupoles

Dispersion and chromaticity