AXEL-2018

Introduction to Particle Accelerators

Review of basic mathematics:

- ✓ *Vectors & Matrices*
- ✓ *Differential equations*

Rende Steerenberg (BE/OP)

5 March 2018

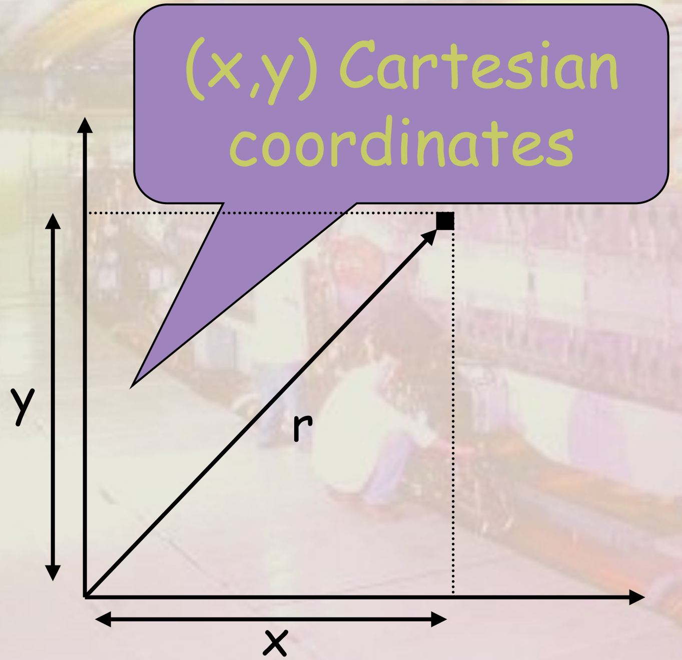
Scalars & Vectors

Scalar, a single quantity or value

Vector, (origin,) length, direction

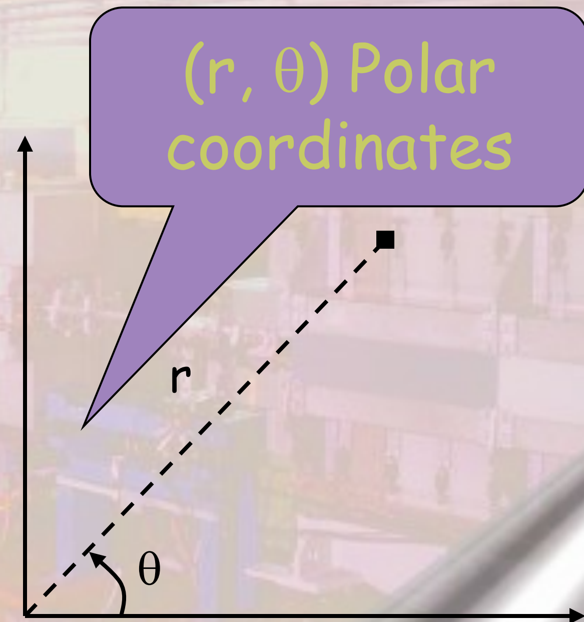
Coordinate Systems

- A scalar is a number: 1,2,..-7, 12.5, etc…..
- A vector has 2 or more quantities associated with it.



r is the length of the vector

$$r = \sqrt{x^2 + y^2}$$

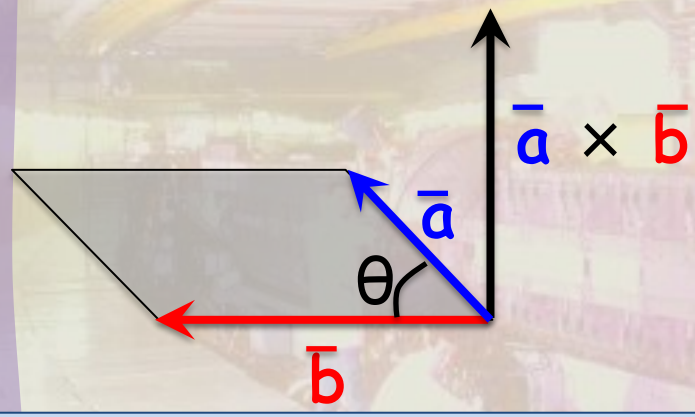


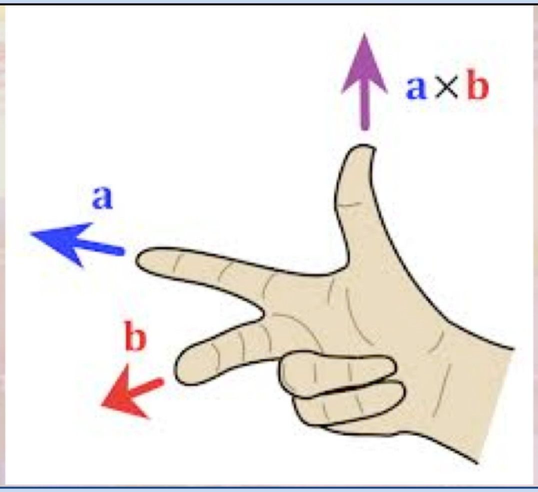
θ gives the direction of the vector

$$\tan(\theta) = \frac{y}{x} \Rightarrow \theta = \arctan\left(\frac{y}{x}\right)$$

Vector Cross Product

$\bar{a}$ and $\bar{b}$ are two vectors in the in a plane separated by angle θ





The cross product $\bar{a} \times \bar{b}$ is defined by:

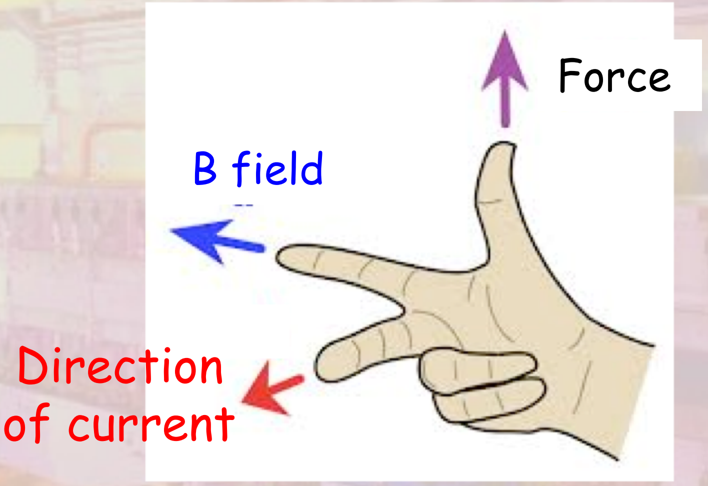
- **Direction**: $\bar{a} \times \bar{b}$ is perpendicular (normal) on the plane through $\bar{a}$ and $\bar{b}$
- The **length** of $\bar{a} \times \bar{b}$ is the surface of the parallelogram formed by $\bar{a}$ and $\bar{b}$

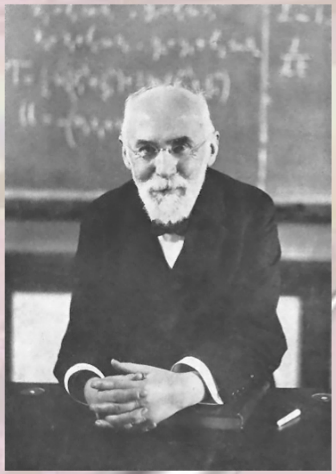
$$\left|\vec{a} \times \vec{b}\right| = \left|\vec{a}\right| \cdot \left|\vec{b}\right| \cdot \sin(\theta)$$

Cross Product & Magnetic Field

The Lorentz force is a pure magnetic field

$$F = e(\vec{v} \times \vec{B})$$

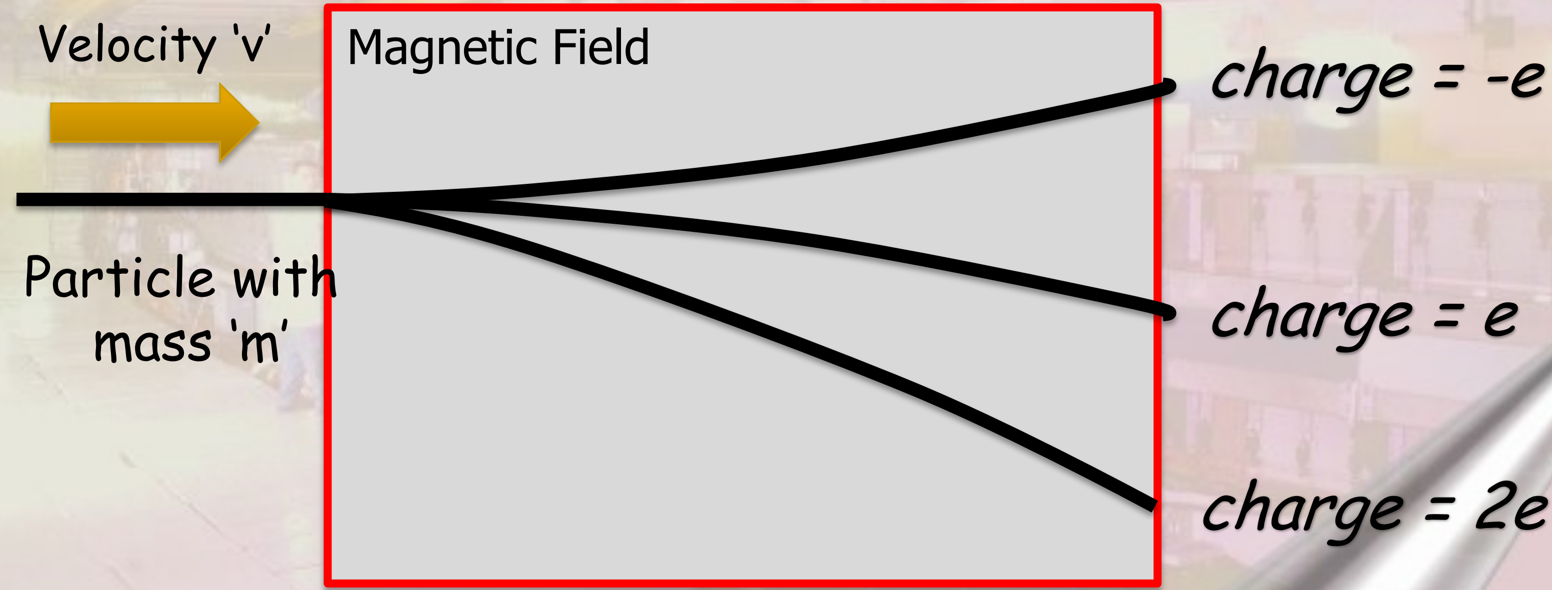




The reason why our particles move around our "circular" machines under the influence of the magnetic fields

Lorentz Force in Action

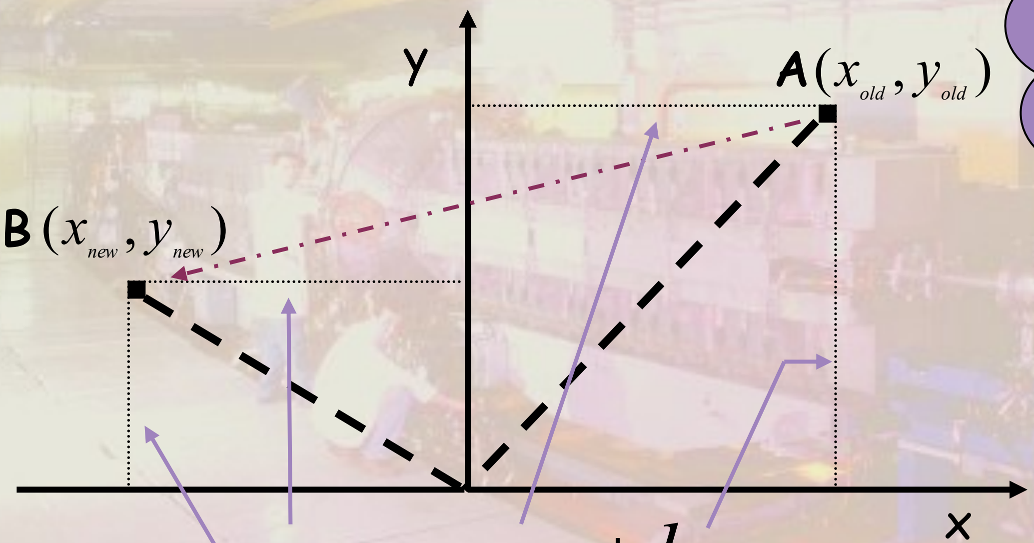
$$F = e(\vec{v} \times \vec{B})$$



The larger the energy of the beam the larger the radius of curvature

Moving for one Point to Another

- To move from one point (A) to any other point (B) one needs control of both <u>Length</u> and <u>Direction</u>.



$$x_{new} = ax_{old} + by_{old}$$

$$y_{new} = cx_{old} + dy_{old}$$

2 equations needed !!!

Defining Matrices (1)

- So, we have:

$$\begin{cases} x_{new} = ax_{old} + by_{old} \\ y_{new} = cx_{old} + dy_{old} \end{cases}$$

- Let's write this as <u>one</u> equation:

$$\overline{B} = M\overline{A}$$

$$\begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix}$$

Rows

Columns

- $\overline{A}$ and $\overline{B}$ are <u>Vectors</u> or <u>Matrices</u>
- $\overline{A}$ and $\overline{B}$ have 2 rows and 1 column
- M is a <u>Matrix</u> and has 2 rows and 2 columns

Defining Matrices (2)

- This means that:

$$\left.\begin{aligned} x_{new} &= ax_{old} + by_{old} \\ y_{new} &= cx_{old} + dy_{old} \end{aligned}\right\} \text{Equals} \left\{ \begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix} \right.$$

- This defines the rules for matrix multiplication.
- More generally we can thus say that…

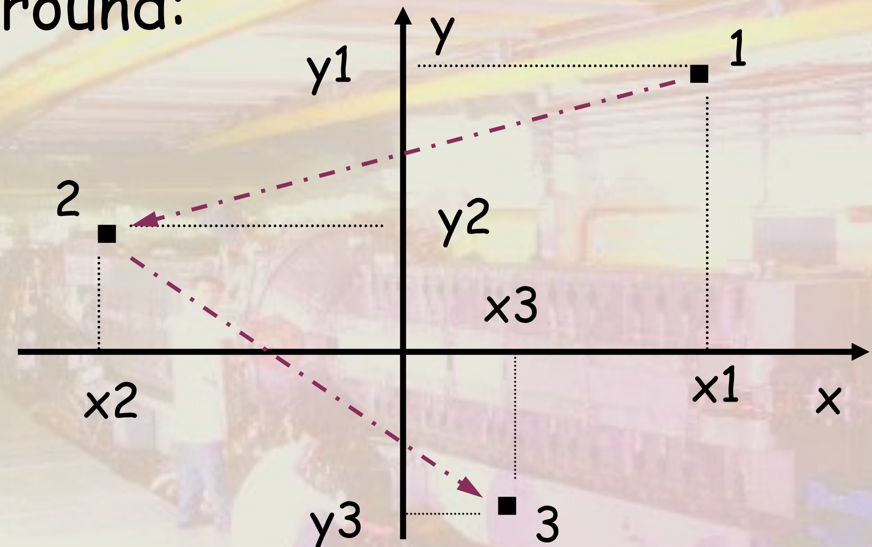
$$\begin{pmatrix} i & j \\ k & l \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

which is be equal to:

$$i = ae + bg, j = af + bh, k = ce + dg, l = cf + dh$$

Applying Matrices

- Let's use what we just learned and move a point around:



- M1 transforms 1 to 2
- M2 transforms 2 to 3
- This defines M3=M2M1

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = M1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = M2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = M2.M1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = M3 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

Matrices & Accelerators

- But… how does this relate to our accelerators ?
- We use matrices to describe the various magnetic elements in our accelerator.
  - The x and y co-ordinates are the <u>position</u> and <u>angle</u> of each individual particle.
  - If we know the position and angle of any particle at one point, then to calculate its position and angle at another point we <u>multiply all the matrices</u> describing the magnetic elements between the two points to give a single matrix
- So, this means that now we are able to calculate the final co-ordinates for any initial pair of particle co-ordinates, provided all the element matrices are known.

Unit Matrix

- There is a special matrix that when multiplied with an initial point will result in the same final point.
- **Unit matrix** :

$$\begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix}$$

- The result is : $\begin{cases} X_{new} = X_{old} \\ Y_{new} = Y_{old} \end{cases}$

- Therefore:

The Unit matrix has no effect on x and y

Going back for one Point to Another

- What if we want to go back from a final point to the corresponding initial point?
- We saw that: $\begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_{old} \\ y_{old} \end{pmatrix}$ or $\overline{B} = M\overline{A}$
- For the reverse we need another matrix M^{-1}

$$\overline{A} = M^{-1}\overline{B}$$

- Combining the two matrices M and M^{-1} we can write:

$$\overline{B} = MM^{-1}\overline{B}$$

- The combination of M and M^{-1} does have no effect thus: $MM^{-1} = Unit\,Matrix$
- M^{-1} is the "inverse" or "reciprocal" matrix of M.

Inverse or Reciprocal Matrix

- If we have: $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,which is a 2 x 2 matrix.
- Then the inverse matrix is calculated by:

$$M^{-1} = \frac{1}{(ad - bc)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- The term (ad – bc) is called the determinate, which is just a number (scalar).

An Accelerator Related Example

- Changing the current in two sets of quadrupole magnets (F & D) changes the horizontal and vertical tunes (Q_h & Q_v).
- This can be expressed by the following matrix relationship:

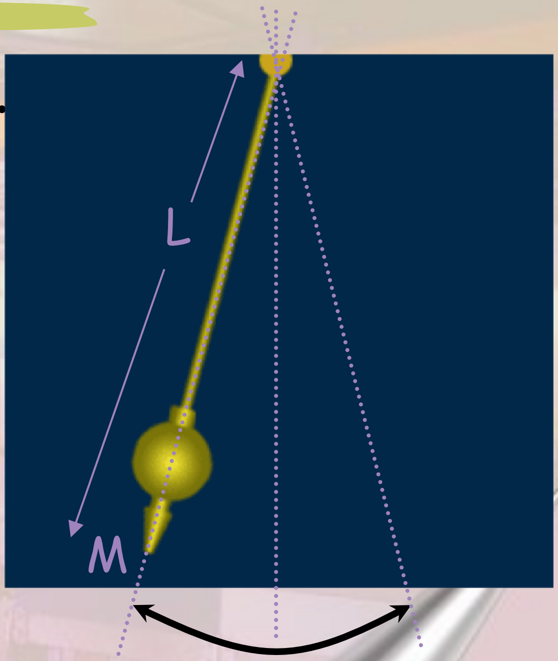
$$\begin{pmatrix} \Delta Q_h \\ \Delta Q_v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \Delta I_F \\ \Delta I_D \end{pmatrix} \quad \text{or} \quad \overline{\Delta Q} = M \overline{\Delta I}$$

- Change I_F then I_D and measure the changes in Q_h and Q_v
- Calculate the matrix M
- Calculate the inverse matrix M^{-1}
- Use now M^{-1} to calculate the current changes (ΔI_F and ΔI_D) needed for any required change in tune (ΔQ_h and ΔQ_v).

$$\overline{\Delta I} = M^{-1} \overline{\Delta Q}$$

Differential Equations

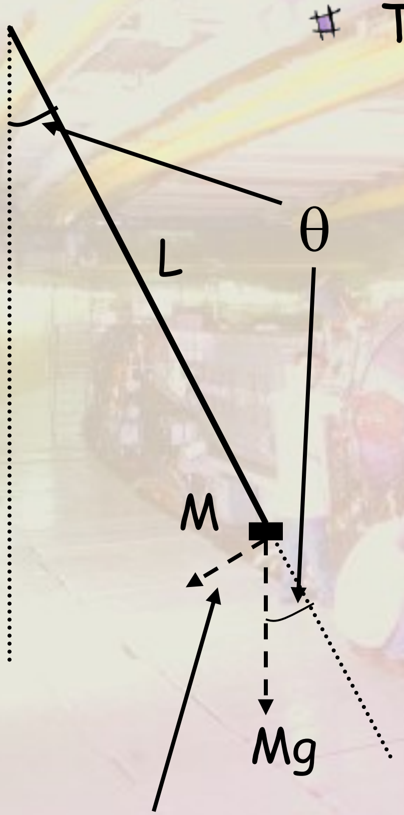
- Let's use a pendulum as an example.



- The length of the Pendulum is L.
- It has mass M attached to it.
- It moves back and forth under the influence of gravity.

- Let's try to find an equation that describes the motion the mass M makes.
- This equation will be a <u>Differential Equation</u>

Establish a Differential Equation



- The <u>distance</u> from the centre = <u>Lθ</u> (since θ is small)
  - The <u>velocity</u> of mass M is: $v = \frac{d(L\theta)}{dt}$
    - The <u>acceleration</u> of mass M is: $a = \frac{d^2(L\theta)}{dt^2}$
      - Newton: Force = mass x acceleration

$$-Mg\sin\theta = M\frac{d^2(L\theta)}{dt^2}$$

Restoring force due to gravity = -Mgsinθ (force opposes motion)

$$\frac{d^2(\theta)}{dt^2} + \left(\frac{g}{L}\right)\theta = 0$$

θ is small
L is constant

Solving the Differential Equation (1)

$$\frac{d^2(\theta)}{dt^2} + \left(\frac{g}{L}\right)\theta = 0$$

- This differential equation describes the motion of a pendulum at small amplitudes.
- Find a solution……Try a good "guess"……

$$\theta = A\cos(\omega t)$$

Oscillation amplitude

- Differentiate our guess (twice)

$$\frac{d(\theta)}{dt} = -A\omega\sin(\omega t)$$

And

$$\frac{d^2(\theta)}{dt^2} = -A\omega^2\cos(\omega t)$$

- Put this and our "guess" back in the original Differential equation.

$$-\omega^2\cos(\omega t) + \left(\frac{g}{L}\right)\cos(\omega t) = 0$$

Solving the Differential Equation (2)

- So we have to find the solution for the following equation:

$$-\omega^2 \cos(\omega t) + \left(\frac{g}{L}\right)\cos(\omega t) = 0$$

- Solving this equation gives: $\omega = \sqrt{\frac{g}{L}}$

- The final solution of our differential equation, describing the motion of a pendulum is as we expected :

$$\theta = A\cos\sqrt{\left(\frac{g}{L}\right)}t$$

Oscillation amplitude (A) — Oscillation frequency ($\sqrt{g/L}$)

Differential Equation & Accelerators

$$\frac{d^2(x)}{dt^2} + (K)x = 0$$

- This is the kind of differential equation that will be used to describe the motion of the particles as they move around our accelerator.
- As we can see, the solution describes:

oscillatory motion

- For any system, where the restoring force is proportional to the displacement, the solution for the displacement will be of the form:

$$x = x_0 \cos(\omega t)$$

- The velocity will be given by:

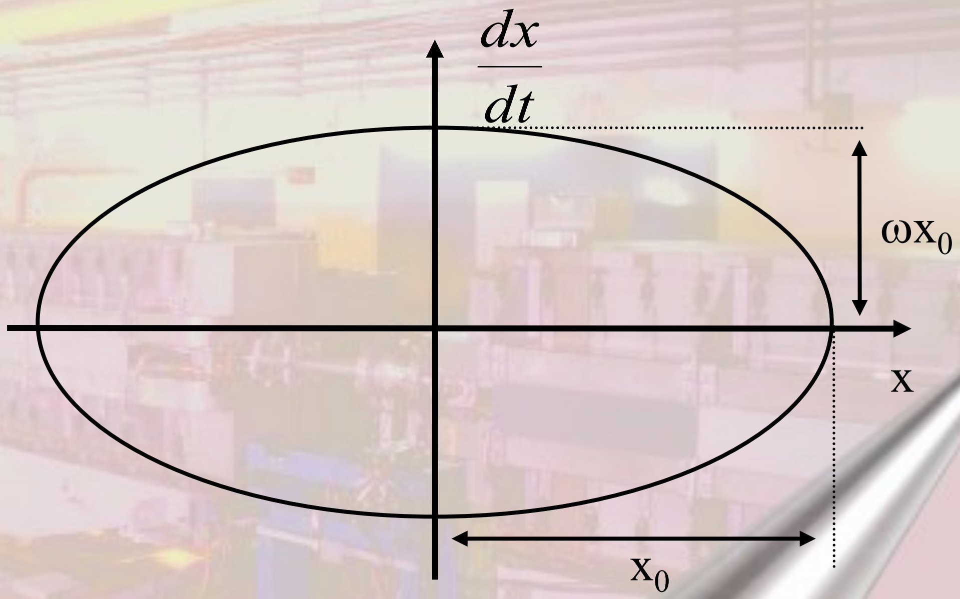
$$\frac{dx}{dt} = -x_0 \omega \sin(\omega t)$$

Visualizing the solution

- Plot the velocity as a function of displacement:

- $x = x_0 \cos(\omega t)$

- $\frac{dx}{dt} = -x_0 \omega \sin(\omega t)$

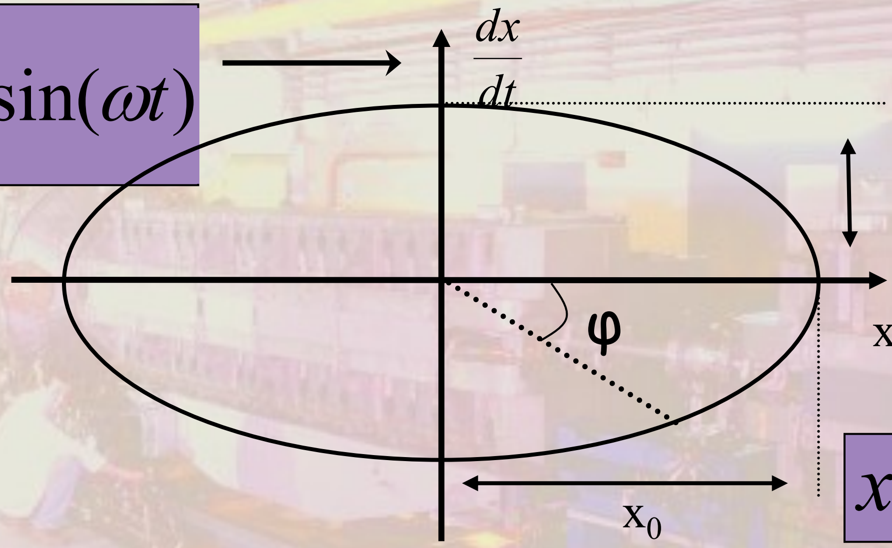


- It is an ellipse.
- As ωt advances by 2 π it repeats itself.
- This continues for (ω t + k 2π), with k=0,±1, ±2,..,..etc

The solution & Accelerators

- How does such a result relate to our accelerator or beam parameters ?

$$\frac{dx}{dt} = -x_0 \omega \sin(\omega t)$$



$$x = x_0 \cos(\omega t)$$

- φ = ωt is called the **phase angle** and the ellipse is drawn in the so called **phase space diagram**.
  - X-axis is normally displacement (position or time).
  - Y-axis is the phase angle or energy.

Questions….,Remarks…?