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Can free strings propagate across plane wave singularities?

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Abstract: We study free string propagation in families of plane wave geometries developing strong scale-invariant singularities in certain limits. We relate the singular limit of the evolution for all excited string modes to that of the center-of-mass motion (the latter existing for discrete values of the overall plane wave profile normalization). Requiring that the entire excitation energy of the string should be finite turns out to be quite restrictive and essentially excludes consistent propagation across the singularity, unless dimensionful scales are introduced at the singular locus (in an otherwise scale-invariant space-time).

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1 Introduction

String propagation in strong gravitational waves has attracted a considerable amount of attention on account of a few highly special properties of such space-times (see [1, 2] among other publications). For one thing, the structure of the curvature tensor in plane gravitational waves implies that these solutions to Einstein's equations (coupled to appropriate matter fields, if necessary) remain uncorrected [1, 3] in a number of higher derivative extensions of general relativity (and, in particular, they do not receive any α'-corrections when introduced as backgrounds in perturbative string theories). Furthermore, the corresponding light cone Hamiltonian of string σ-models turns out to be quadratic and admits a fairly thorough analytic treatment. (This class of backgrounds also admits a natural formulation of the matrix theory description of quantum gravity [4, 5].)

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In this publication, we shall concentrate on backgrounds of the following form (so-called exact plane waves):

$$ds^2 = -2dX^+dX^- - F(X^+)\sum_{i=1}^{d}(X^i)^2(dX^+)^2 + \sum_{i=1}^{d}(dX^i)^2. \tag{1.1}$$

This representation of the metric is often called the Brinkmann form. The case of constant $F(X^+)$ corresponds to supersymmetric plane waves studied in [6], and it is quite different from the the rapidly varying $F(X^+)$ we intend to consider. (A coordinate transformation can be performed into the so-called Rosen coordinates eliminating the dependence of the metric on the transverse coordinates X^i. The resulting metric depends on X^+ only and displays manifestly a plane-fronted space-time wave propagating at the speed of light. However, the Rosen parametrization tends to suffer from coordinate singularities, and we shall work with the Brinkmann form.)

The function $F(X^+)$ contained in (1.1) is completely arbitrary, and one may ask, for example, what happens to quantum strings propagating in such space-times when the wave profile $F(X^+)$ develops an isolated singularity. This question is of some interest per se, since studies of string theory in the presence of space-time singularities have played a pivotal role in the development of the subject (and, in this particular case, we are dealing with singularities in time-dependent backgrounds). Additional heuristic justification for our studies is provided by the observation that plane waves of the type (1.1) with

$$F(X^+) \sim \frac{1}{(X^+)^2} \tag{1.2}$$

arise as Penrose limits of a broad class [7] of space-time singularities (including the Friedmann-Lemaître-Robertson-Walker cosmological singularities). With $F(X^+)$ of (1.2), the metric (1.1) is invariant under scaling transformations $X^+ \to \alpha X^+$, $X^- \to X^-/\alpha$ (identical to Lorentz boosts in flat space-time). Note that this type of singularities is considerably stronger than the so-called "weak singularities" of [8].

Free string propagation on (1.1) with $F(X^+)$ given by (1.2) has been previously studied in [2]. In particular, it was suggested in that publication that the question of propagation across the $1/(X^+)^2$ singularity in the metric can be addressed by employing analytic continuation in the complex X^+-plane. We believe that this issue merits further elucidation.

In the context of string theory and related approaches to quantum gravity, there is a general expectation that the space-time background used for formulating the theory should satisfy some stringent consistency conditions. For perturbative string theories, these conditions take the form of the appropriate supergravity equations of motion together with an infinite tower of α'-corrections. For non-singular plane waves, all the α'-corrections vanish automatically on account of the special properties of the Riemann tensor corresponding to these space-times. For singular space-times, the question of background consistency conditions *at* the singular point appears to be extremely subtle. Indeed, what should replace the supergravity equations of motion at the singular point where they obviously break down? Ad hoc prescriptions are not likely to produce meaningful results under these circumstances.

One approach to formulating string theory in backgrounds (1.1)–(1.2) is to resolve the singular plane wave profile into a non-singular function, perform the necessary computations and see if the result has a meaningful singular limit. (This approach was advocated in [1], where a conjecture was made that for certain choices of the plane wave profile, taking a singular limit may result in well-defined transition amplitudes. We intend to consider this question quantitatively.) Note that, for the resolved space-times of this sort, perturbative string background consistency conditions are automatically satisfied to all orders in α'. The only non-trivial question is the existence of a singular limit.

But how should one resolve? We want to construct a function $F(X^+, \epsilon)$ in such a way that

$$\lim_{\epsilon \to 0} F(X^+, \epsilon) = \frac{\text{const}}{(X^+)^2} \tag{1.3}$$

everywhere away from $X^+ = 0$. There is in principle a large amount of ambiguity associated with such resolutions. One class appears to be very special however. The background (1.1)–(1.2) possesses a scaling symmetry and does not depend on any dimensionful parameters. It is natural to demand that this symmetry should be recovered when the resolution is removed. This will happen if the resolved profile $F(X^+, \epsilon)$ does not depend on any dimensionful parameters other than the resolution parameter ϵ. In this case, on dimensional grounds,

$$F(X^+, \epsilon) = \frac{\lambda}{\epsilon^2} \Omega(X^+/\epsilon). \tag{1.4}$$

The limit (1.3) will be recovered if

$$\Omega(\eta) \to \frac{k}{\eta^2} + O\left(\frac{1}{\eta^b}\right) \tag{1.5}$$

for large values of η, with some $b > 2$. Note that the fact that the original background possesses a certain symmetry (away from $X^+ = 0$!) in no way implies that we *must* resolve in a way consistent with this symmetry. For resolved profiles different from (1.4), the limit of the metric may still be given by (1.3) away from $X^+ = 0$ (and thus be scale invariant), but additional dimensionful scales may become buried inside the singularity at $X^+ = 0$ (in a way that only affects processes involving singularity crossing). One would need some strong physical rationale for introducing such scales buried at the singular locus, and in the present publication we shall simply study the "scale-invariant" resolutions (1.4).

The structure of the paper is as follows: we will first derive the Hamiltonian for a free string in the background (1.1)–(1.2). Then we recapitulate the main results of [9] for the evolution of the center-of-mass motion across the plane wave singularity. We extend this analysis to the evolution of excited string modes. We conclude by discussing stringent conditions arising if one demands the total mass of the string to remain finite after it crosses the singularity.

2 Free strings in plane waves

Due to the presence of covariantly constant null vectors in plane wave geometries, the string theory σ-model can be analyzed explicitly in such backgrounds, and reduces to a set of independent classical time-dependent harmonic oscillators. In this section, we re-state this familiar material in a way convenient for our present investigations.

2.1 The light cone gauge

String worldsheet fermions are free in plane wave backgrounds [10]. We shall therefore concentrate on the bosonic part of the string action, given by

$$I = -\frac{1}{4\pi\alpha'}\int d\tau \int_0^{2\pi} d\sigma \sqrt{-g}\left(g^{ab}G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu - \frac{1}{2}\alpha' R^{(2)}\Phi\right). \tag{2.1}$$

We choose light-cone gauge $X^+ = \alpha' p^+ \tau$ and gauge-fix the metric,

$$\det(g_{ab}) = -1, \qquad \partial_\sigma g_{\sigma\sigma} = 0, \tag{2.2}$$

to obtain the following Lagrangian, where we have solved for $g_{\tau\tau}$:

$$\begin{aligned} L = -\frac{1}{4\pi\alpha'}\int_0^{2\pi} \mathrm{d}\sigma \Bigg(& 2g_{\sigma\sigma}p^+\alpha'\partial_\tau X^- - g_{\sigma\sigma}\sum_{i=1}^{8}\left((\partial_\tau X^i)^2 + \frac{(\alpha' p^+)^2}{\epsilon^2}\Omega(\alpha' p^+\tau/\epsilon)(X^i)^2\right) \\ & -2g_{\tau\sigma}\left(\alpha' p^+\partial_\sigma X^- - \partial_\tau X^i \partial_\sigma X^i\right) + g_{\sigma\sigma}^{-1}(1-g_{\tau\sigma}^2)\sum_{i=1}^{8}(\partial_\sigma X^i)^2 - \frac{1}{2}\alpha' R^{(2)}\Phi\Bigg). \end{aligned} \tag{2.3}$$

We rescale $\epsilon = \epsilon'\alpha' p^+$,

$$\frac{(\alpha' p^+)^2}{\epsilon^2}\Omega(X^+/\epsilon) = \frac{1}{\epsilon'^2}\Omega(\tau/\epsilon'), \tag{2.4}$$

and from here on, we will denote worldsheet time $\tau = t$ and write ϵ instead of ϵ'. The σ-dependent part of the oscillator X^- is non-dynamical and enforces $g_{\tau\sigma} = 0$. The σ-independent part of the oscillator X^- can be eliminated as a constraint ($g_{\sigma\sigma} = 1$), the (dynamically non-trivial) coupling to the dilaton disappears (see, e.g., [2]), and we can write the following worldsheet Hamiltonian

$$H = \frac{1}{4\pi\alpha'}\int \mathrm{d}\sigma \sum_{i=1}^{d}\left(\pi^2(P_i)^2 + \frac{\lambda}{\epsilon^2}\Omega(\tau/\epsilon)(X^i)^2 + \left(\partial_\sigma X^i\right)^2\right), \tag{2.5}$$

where P_i are momenta conjugate to X^i. We will now choose units in which $\alpha' = 1$. If we Fourier transform the σ-coordinate,

$$X^i(t,\sigma) = X_0^i(t) + \sqrt{2}\sum_{n>0}\left(\cos(n\sigma)\, X_n^i(t) + \sin(n\sigma)\, \tilde{X}_n^i(t)\right), \tag{2.6}$$

we obtain a set of time-dependent harmonic oscillator Hamiltonians

$$H = \sum_{n=0}^{\infty}\sum_{i=1}^{d} H_{\mathrm{ni}}, \tag{2.7}$$

$$H_{0i} = \frac{(P_{0i})^2}{2} + \frac{\lambda}{\epsilon^2}\Omega(t/\epsilon)\frac{(X_0^i)^2}{2}, \tag{2.8}$$

$$H_{\mathrm{ni}} = \frac{(P_{\mathrm{ni}})^2 + (\tilde{P}_{\mathrm{ni}})^2}{2} + \left(n^2 + \frac{\lambda}{\epsilon^2}\Omega(t/\epsilon)\right)\frac{(X_n^i)^2 + (\tilde{X}_n^i)^2}{2}. \tag{2.9}$$

2.2 WKB solution for time-dependent harmonic oscillator

The Hamiltonian (2.7) is quadratic and the solution to the corresponding Schrödinger equation,

$$i\frac{\partial}{\partial t}\phi(t;X_n^i) = \left(\sum_n \sum_{i=1}^{d} H_{\rm ni}\right)\phi(t;X_n^i), \tag{2.10}$$

can be found using WKB techniques, which are exact for quadratic Hamiltonians. From (2.10) it follows that

$$i\frac{\partial}{\partial t}\phi_n^i(t;X_n^i) = -\frac{1}{2}\frac{\partial^2}{(\partial X_n^i)^2}\phi_n^i(t;X_n^i) + \frac{1}{2}\left(n^2+\frac{\lambda}{\epsilon^2}\Omega(t/\epsilon)\right)\left(X_n^i\right)^2\phi_n^i(t;X_n^i), \tag{2.11}$$

if we separate variables as

$$\phi(t;\mathbf{X}) = \prod_n \prod_{i=1}^{8}\phi_n^i(t;X_n^i). \tag{2.12}$$

We then take the WKB ansatz

$$\phi_n^i(t;X) = \mathcal{A}_n(t_1,t)\exp\left(iS_{cl;n}[X_{1,n}^i,t_1|X_n^i,t]\right), \tag{2.13}$$

where $S_{cl;n}[X_{1,n}^i,t_1|X_n^i,t]$ is the "classical action" evaluated for the path going from $X_{1,n}^i$ at the time t_1 to X_n^i at the time t,

$$S_{\rm cl}[X_{1,n}^i,t_1|X_n^i,t] = \int_{t_1}^{t} {\rm d}t'\left(\frac{(\dot{X}_n^i)^2}{2} - \left(n^2+\frac{\lambda}{\epsilon^2}\Omega\left(\frac{t'}{\epsilon}\right)\right)\frac{(X_n^i)^2}{2}\right). \tag{2.14}$$

If $\mathcal{A}_n(t_1,t)$ satisfies

$$-2\frac{\partial}{\partial t}\mathcal{A}_n(t_1,t) = \mathcal{A}_n(t_1,t)\frac{\partial^2}{\partial (X_n^i)^2}S_{\rm cl}[X_{1,n}^i,t_1|X_n^i,t], \tag{2.15}$$

then (2.13) satisfies the original Schrödinger equation exactly.

Up to normalization, a basis of solutions, labelled by the initial condition $X_n^i(t_1) = X_{1,n}^i$, is given by [9],

$$\phi(t;X_n^i) \sim \prod_{\rm ni}\frac{1}{\sqrt{\mathcal{C}(t_1,t)}}\exp\left(-\frac{i}{2\mathcal{C}}\sum_{i=1}^{d}\left[(X_{1,n}^i)^2\partial_{t_1}\mathcal{C} - (X_n^i)^2\partial_{t_2}\mathcal{C} + 2X_{1,n}^iX_n^i\right]\right), \tag{2.16}$$

where $\mathcal{C}(t_1,t_2)$ (suppressing the index n) is a solution to the "classical equation of motion" for the time-dependent harmonic oscillator Hamiltonian (2.9):

$$\partial_{t_2}^2\mathcal{C}(t_1,t_2) + \left(n^2+\frac{\lambda}{\epsilon^2}\,\Omega(t_2/\epsilon)\right)\mathcal{C}(t_1,t_2) = 0, \tag{2.17}$$

with initial conditions specified as

$$\mathcal{C}(t_1,t_2)|_{t_1=t_2} = 0, \qquad \partial_{t_2}\mathcal{C}(t_1,t_2)|_{t_1=t_2} = 1. \tag{2.18}$$

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We shall refer to $\mathcal{C}(t_1, t_2)$ as "compression factor", since it describes convergence of solutions to the corresponding harmonic oscillator equation starting at the same point at the moment t_1. (If $\mathcal{C}(t_1, t_2)$ vanishes, then t_2 is a focal point, as the difference between any two solutions with the same initial position $X(t_1)$ is proportional to $\mathcal{C}(t_1, t_2)$.) A useful representation of $\mathcal{C}(t_1, t_2)$ is given by

$$\mathcal{C}(t_1, t_2) = \frac{f(t_1)h(t_2) - f(t_2)h(t_1)}{W[f, h]}, \tag{2.19}$$

where $f(t)$ and $h(t)$ are two independent solutions to the differential equation under consideration, and the Wronskian W is given by

$$W[f, h] = f\dot{h} - h\dot{f}. \tag{2.20}$$

To derive the singular limit of the wavefunction (2.16) it is sufficient to study the singular limit of (2.17)–(2.18).

3 The singular limit for the center-of-mass motion

For the $n = 0$ mode, we obtain as the "classical equation of motion"

$$\ddot{X} + \frac{\lambda}{\epsilon^2}\Omega(t/\epsilon)X = 0. \tag{3.1}$$

We need to study the $\epsilon \to 0$ limit of the solution that obeys the initial conditions

$$X(t_1) = 0, \qquad \dot{X}(t_1) = 1, \qquad t_1 < 0. \tag{3.2}$$

The singular limit of solutions to this equation has been analyzed in [9]. Performing a scale transformation $Y(\eta) = X(\eta\epsilon)$, with $\eta = t/\epsilon$, removes the ϵ-dependence from the equation, leaving

$$\frac{\partial^2}{\partial\eta^2}Y + \lambda\Omega(\eta)Y = 0. \tag{3.3}$$

This scale transformation is possible because our initial singular metric was scale-invariant and we have resolved it as in (1.4) without introducing any dimensionful parameters besides ϵ. The existence of a singular limit is then translated [9] into constraints on on the asymptotic behavior of solutions to (3.3). These "boundary conditions at infinity" are strongly reminiscent of a Sturm-Liouville problem, and it is natural that a discrete spectrum of λ is singled out by imposing the existence of a singular limit.

For the specific asymptotics of our resolved profile (1.5), it can be shown [9] that, in the infinite past and infinite future, the solutions approach a linear combination of two powers (denoted below a and $1 - a$, with a being a function of $k\lambda$, cf. (1.4)–(1.5)). This power law behavior simply corresponds to the regime when the second term on the right hand side of (1.5) can be neglected compared to the first. It is then convenient to form two bases of solutions, one asymptotically approaching the two powers (dominant and subdominant) at $\eta \to -\infty$,

$$Y_{1-}(\eta) = |\eta|^{a_-} + o(|\eta|^{a_-}), \qquad Y_{2-}(\eta) = |\eta|^{1-a_-} + o(|\eta|^{1-a_-}), \tag{3.4}$$

and another behaving similarly at $\eta \to +\infty$

$$Y_{1+}(\eta) = |\eta|^{a_+} + o(|\eta|^{a_+}), \qquad Y_{2+}(\eta) = |\eta|^{1-a_+} + o(|\eta|^{1-a_+}), \tag{3.5}$$

where $a_\pm$ is given by

$$a_\pm = \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda k_\pm}. \tag{3.6}$$

(We are temporarily assuming that k can take two different values $k_\pm$ for the positive and negative time asymptotics, a possibility that will be discarded shortly.) The two bases need, of course, to be related by a linear transformation:

$$\begin{bmatrix} Y_{1-}(\eta) \\ Y_{2-}(\eta) \end{bmatrix} = Q(\lambda) \begin{bmatrix} Y_{1+}(\eta) \\ Y_{2+}(\eta) \end{bmatrix}, \tag{3.7}$$

where $Q(\lambda)$ is a 2×2 matrix whose determinant is constrained by Wronskian conservation as

$$W[Y_{1-}, Y_{2-}] = W[Y_{1+}, Y_{2+}] \det Q. \tag{3.8}$$

The singular limit has been rigorously considered in [9], but the results can be understood heuristically from the following argument. Imagine one is trying to construct a solution $\tilde{Y}$ to (3.3) satisfying some (ϵ-independent) initial conditions at $\eta_1 = t_1/\epsilon < 0$. This solution can be expressed in terms of Y_{1-} and Y_{2-} (a complete basis) as

$$\tilde{Y} = C_1 Y_{1-} + C_2 Y_{2-}. \tag{3.9}$$

Since the initial conditions are specified at $\eta_1 = t_1/\epsilon$, the asymptotic expansions (3.4) are valid. There needs to be a non-trivial contribution from both Y_{1-} and Y_{2-} in the above formula in order to satisfy general initial conditions. Hence, the two terms on the right hand side should be of order 1. Therefore, we should have

$$C_1 = O(\epsilon^{a_-}), \qquad C_2 = O(\epsilon^{1-a_-}). \tag{3.10}$$

If we now apply (3.7) and (3.5) to evaluate $\tilde{Y}$ at a large positive $\eta = t_2/\epsilon$, the powers of ϵ in C_1 and C_2 will combine with the powers of ϵ originating from Y_{1+} and Y_{2+} and yield

$$\begin{aligned} \tilde{Y}(t_2/\epsilon) =& Q_{11}(\lambda) t_2^{a_+} O(\epsilon^{a_- - a_+}) + Q_{12}(\lambda) t_2^{1-a_+} O(\epsilon^{a_- + a_+ - 1}) \\ &+ Q_{21}(\lambda) t_2^{a_+} O(\epsilon^{1-a_- - a_+}) + Q_{22}(\lambda) t_2^{1-a_+} O(\epsilon^{a_+ - a_-}). \end{aligned} \tag{3.11}$$

Since a_+ and a_- are greater than $1/2$, this expression can only have an $\epsilon \to 0$ limit if $a_+ = a_-$ (i.e., $k_+ = k_-$ and we can set both equal to 1 by redefining λ) and $Q_{21}(\lambda) = 0$. The latter condition implies that the absolute normalization λ of the plane wave profile $\Omega(\eta)$ will generically lie in a discrete spectrum, dependent on the specific way the singularity is resolved, i.e., the shape of $\Omega(\eta)$. A particular exactly solvable example for this discrete spectrum (there called "light-like reflector plane") has been given in [11]. With $Q_{21}(\lambda) = 0$ and $\det Q = -1$, the matrix Q can be written as

$$Q = \begin{bmatrix} q & \tilde{q} \\ 0 & -1/q \end{bmatrix}, \tag{3.12}$$

with q being a real nonzero number ($\tilde{q}$ does not affect the singular limit). For flat space-time we have $q = 1$ and for the "light-like reflector plane" of [11] we have $q = -1$. In the singular limit, a basis of solutions is given by

$$\begin{aligned} &Y_1(t) = (-t)^a, && Y_2(t) = (-t)^{1-a}, && t < 0, \\ &Y_1(t) = q\, t^a, && Y_2(t) = -\frac{1}{q} t^{1-a}, && t > 0. \end{aligned} \tag{3.13}$$

4 The singular limit for excited string modes

Following our general discussion of free strings in plane wave backgrounds, the evolution of excited string modes is described by time-dependent harmonic oscillator equations

$$\frac{\partial^2}{\partial t^2} X(t) + \left(n^2 + \frac{\lambda}{\epsilon^2} \Omega(t/\epsilon) \right) X(t) = 0. \tag{4.1}$$

Solutions for the wavefunctions of the excited string modes can be expressed in terms of a particular solution to this equation $\mathcal{C}(t_1, t_2)$ defined by (2.17)–(2.18). Hence, to analyze the singular ($\epsilon \to 0$) limit of the excited modes dynamics, it should suffice to analyze the singular limit of $\mathcal{C}(t_1, t_2)$. Because n^2 is finite, it is natural to expect that it does not affect the existence of the singular limit (governed by the singularity emerging from $\Omega(t/\epsilon)$). We shall prove that it is indeed the case for positive λ (for negative λ unstable motion of the inverted harmonic oscillator leads to divergences[1]).

To derive $\mathcal{C}(t_1, t_2)$ for equation (4.1) we use the following strategy: the differential equation (4.1) is linear and any solution $X(t_2)$ at $t = t_2$ can be written in terms of a "transfer matrix" T that only depends on the initial and final times,

$$\begin{bmatrix} X(t_2) \\ \dot{X}(t_2) \end{bmatrix} = T(t_1, t_2) \begin{bmatrix} X(t_1) \\ \dot{X}(t_1) \end{bmatrix}. \tag{4.2}$$

The transfer matrix can be expressed as

$$T(t_1, t_2) = \begin{bmatrix} -\partial_{t_i} \mathcal{C}(t_1, t_2) & \mathcal{C}(t_1, t_2) \\ -\partial_{t_i} \partial_{t_f} \mathcal{C}(t_1, t_2) & \partial_{t_f} \mathcal{C}(t_1, t_2) \end{bmatrix}, \tag{4.3}$$

where ∂_{t_i} and ∂_{t_f} indicate differentiation with respect to the first and second argument respectively. The transfer matrix is completely determined once $\mathcal{C}(t_1, t_2)$ has been determined, and vice versa. We will now use the fact that transfer matrices of subintervals are combined by ordinary matrix multiplication. Dividing the solution region into three sub-intervals, we shall calculate the transfer matrices T_k for each sub-interval k and apply multiplication to construct the total transfer matrix. The sub-intervals shall be chosen as indicated in figure 1. We use t_ϵ to indicate a time that will approach zero in the singular

[1] More specifically, the divergences arise from sub-leading infinities in the position of the inverted harmonic oscillator, while the leading infinities cancel. (Such sub-leading infinities are absent for the center-of-mass motion analyzed in section 3, hence no analogous divergences in that case.) Further details are given in section 4.4 and appendix A.

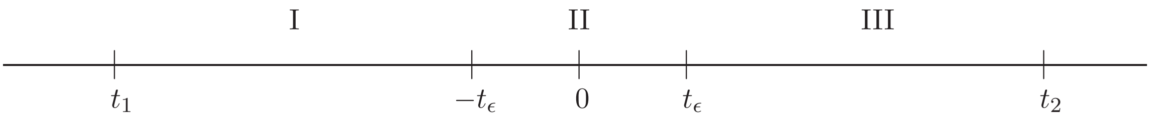

Figure 1. The solution region of (4.1) divided into three sub-intervals.

limit as

$$t_\epsilon = \epsilon^{1-c}\tilde{t}^c, \tag{4.4}$$

with $\tilde{t}$ staying finite in relation to the "moments of observation" t_1 and t_2. The number c (between 0 and 1) will be chosen later as needed for our proof. On each interval, we can write the transfer matrix T_k in terms of the "compression factor" $\mathcal{C}_k$. Using matrix multiplication to construct the full transfer matrix T, we can now deduce an expression for the "compression factor" of the complete interval

$$\begin{aligned}\mathcal{C}(t_1,t_2) =& \mathcal{C}_I(t_1,-t_\epsilon)\partial_{t_i}\mathcal{C}_{\mathrm{II}}(-t_\epsilon,t_\epsilon)\partial_{t_i}\mathcal{C}_{\mathrm{III}}(t_\epsilon,t_2) - \partial_{t_f}\mathcal{C}_I(t_1,-t_\epsilon)\mathcal{C}_{\mathrm{II}}(-t_\epsilon,t_\epsilon)\partial_{t_i}\mathcal{C}_{\mathrm{III}}(t_\epsilon,t_2)\\ &- \mathcal{C}_I(t_1,-t_\epsilon)\partial_{t_i}\partial_{t_f}\mathcal{C}_{\mathrm{II}}(-t_\epsilon,t_\epsilon)\mathcal{C}_{\mathrm{III}}(t_\epsilon,t_2) + \partial_{t_f}\mathcal{C}_I(t_1,-t_\epsilon)\partial_{t_f}\mathcal{C}_{\mathrm{II}}(-t_\epsilon,t_\epsilon)\mathcal{C}_{\mathrm{III}}(t_\epsilon,t_2),\end{aligned} \tag{4.5}$$

in terms of the "compression factors" of the three sub-intervals. Once again, ∂_{t_i} and ∂_{t_f} differentiate $\mathcal{C}$ with respect to its first and second argument (initial and final time).

To study the existence of the singular limit of $\mathcal{C}(t_1,t_2)$, we will use the following strategy: for two linear differential equations related by a small perturbation we will establish a bound on the difference between perturbed and unperturbed solutions with the same initial conditions. This bound will, of course, apply to $\mathcal{C}_k$. For each of the three sub-intervals introduced above, we will consider a simplified differential equation that is a good approximation to equation (4.1) on the corresponding interval:

- Region I and III: $\ddot{X}(t) + \left(n^2 + \lambda/t^2\right)X(t) = 0$ (related to Bessel's equation);
- Region II: $\ddot{X}(t) + \lambda/\epsilon^2\Omega(t/\epsilon)X(t) = 0$ (equation of motion for the zero mode).

Then, on each sub-interval, $\mathcal{C}_k$ can be written as the sum of a simplified "compression factor" $\bar{\mathcal{C}}_k$ satisfying the simplified differential equation on this sub-interval, plus a small perturbation $\delta\mathcal{C}_k$. We will prove that, in the singular limit, the $\delta\mathcal{C}_k$ will drop out of the expression for the total "compression factor" $\mathcal{C}(t_1,t_2)$.

Most of this section is dedicated to implementing the proof we have just outlined. The reader primarily interested in the discussion of the singular limit and content with the general sketch given above can skip to section 4.4.

4.1 Bounds on solutions to perturbed differential equations

In view of the subsequent application to the singular limit analysis, we would like to bound the difference δX between the solution $X(t)$ of a perturbed differential equation,

$$\frac{\partial^2}{\partial t^2}X + (\Upsilon + \delta\Upsilon)\,X = 0, \tag{4.6}$$

and the solution $\bar{X}(t)$ of an unperturbed differential equation,

$$\frac{\partial^2}{\partial t^2}\bar{X} + \Upsilon\bar{X} = 0, \tag{4.7}$$

where we take

$$X = \bar{X} + \delta X \tag{4.8}$$

and demand that the initial conditions remain unchanged:

$$X(t_0) = \bar{X}(t_0), \qquad \partial_t X(t_0) = \partial_t \bar{X}(t_0). \tag{4.9}$$

If we substitute (4.8) and (4.7) into (4.6) we obtain a differential equation for the perturbation on the solution.

$$\frac{\partial^2}{\partial t^2}\delta X + \Upsilon(t)\delta X = -\delta\Upsilon(t)\left(\bar{X} + \delta X\right). \tag{4.10}$$

A formal solution to (4.10) is given by

$$\delta X(t) = -\int_{-\infty}^{\infty} G_r(t,t')\delta\Upsilon(t')\left(\bar{X}(t') + \delta X(t')\right)dt', \tag{4.11}$$

with the Green function $G_r(t,t')$ satisfying

$$\left(\frac{\partial^2}{\partial t^2} + \Upsilon(t)\right)G_r(t,t') = \delta(t-t') \tag{4.12}$$

and initial conditions

$$G_r(t,t')|_{t=t_0} = 0, \qquad \partial_t G_r(t,t')|_{t=t_0} = 0. \tag{4.13}$$

Therefore, we can write the Green function in terms of the "compression factor" $\bar{\mathcal{C}}$ of the unperturbed equation (4.7), where $\bar{\mathcal{C}}$ obeys the same initial conditions as in (2.18):

$$G_r(t,t') = \begin{cases} \bar{\mathcal{C}}(t',t) & t_0 < t' < t, \\ 0 & \text{otherwise.} \end{cases} \tag{4.14}$$

To obtain a bound on δX we will invoke the so-called Gronwall inequality [12].

4.1.1 The Gronwall inequality

Let $I = [A,B]$. Assume β and α real valued and continuous on I and $\beta \geq 0$. If u is continuous, real valued on I and satisfies the integral inequality

$$u(t) < \alpha(t) + \int_A^t \beta(s)u(s)ds, \qquad t \in I, \tag{4.15}$$

then

$$u(t) < \alpha(t) + \int_A^t \beta(s)\alpha(s)\exp\left(\int_s^t \beta(r)dr\right)ds, \qquad t \in I. \tag{4.16}$$

Proof. First we define

$$z(t) = \int_A^t \beta(s)u(s)\mathrm{d}s, \qquad t \in I. \tag{4.17}$$

Then, after differentiation and using the initial assumption (4.15), we obtain

$$z'(t) = \beta(t)u(t) \leq \beta(t)\alpha(t) + \beta(t)z(t). \tag{4.18}$$

Using the line above we write

$$\left[\exp\left(-\int_A^s \beta(u)\mathrm{d}u\right) z(s)\right]' = \exp\left(-\int_A^s \beta(r)\mathrm{d}r\right)\left(z'(s) - \beta(s)z(s)\right) \tag{4.19}$$

$$\leq \beta(s)\alpha(s)\exp\left(-\int_A^s \beta(u)\mathrm{d}u\right) \qquad s \in I. \tag{4.20}$$

We integrate from a to t and obtain,

$$\exp\left(-\int_A^t \beta(s)\mathrm{d}s\right) z(t) \leq \int_A^t \beta(s)\alpha(s)\exp\left(-\int_A^s \beta(u)\mathrm{d}u\right)\mathrm{d}s \qquad t \in I. \tag{4.21}$$

From assumption (4.15) and (4.21) we now derive the desired inequality,

$$u(t) \leq \alpha(t) + z(t) \leq \alpha(t) + \exp\left(\int_A^t \beta(r)\mathrm{d}r\right)\int_A^t \beta(s)\alpha(s)\exp\left(-\int_A^s \beta(u)\mathrm{d}u\right)\mathrm{d}s \tag{4.22}$$

$$= \alpha(t) + \int_A^t \beta(s)\alpha(s)\exp\left(\int_s^t \beta(u)\mathrm{d}u\right)\mathrm{d}s, \qquad t \in I. \tag{4.23}$$

4.1.2 Bounds on the perturbations δX

From (4.11) we derive the following bound on the formal solution δX

$$|\delta X(t)| < \int_{-\infty}^{\infty} |G_r(t,t')\delta\Upsilon(t')\bar{X}(t')|dt' + \int_{-\infty}^{\infty} |G_r(t,t')\delta\Upsilon(t')\delta X(t')|dt'. \tag{4.24}$$

We will now use the fact that, by virtue of (4.14), where nonzero, $G_r(t,t') = C(t',t)$. Hence (cf. (2.19)), inside the integral,

$$|G(t,t')| < \frac{1}{|W|}\left(|f|_M|h(t')| + |f(t')||h|_M\right) \equiv g(t'), \tag{4.25}$$

with $|f|_M$ and $|h|_M$ being the absolute value maxima of these functions on the integration domain. The integration regions are in fact finite, since (4.14) vanishes unless $t_0 < t' < t$:

$$|\delta X(t)| < \int_{t_0}^{t} |g(t')\delta\Upsilon(t')\bar{X}(t')|dt' + \int_{t_0}^{t} |g(t')\delta\Upsilon(t')\delta X(t')|dt'. \tag{4.26}$$

Since $g(t')$ is independent of t we can now apply Gronwall's inequality to obtain

$$|\delta X(t)| < \int_{t_0}^{t} |g(t')\delta\Upsilon(t')\bar{X}(t')|dt'$$

$$+ \int_{t_0}^{t}\left(\int_{t_0}^{t'} |g(t'')\delta\Upsilon(t'')\bar{X}(t'')|dt''\right)|g(t')\delta\Upsilon(t')|\exp\left(\int_{t'}^{t} |g(t'')\delta\Upsilon(t'')|dt''\right)dt'. \tag{4.27}$$

On the interval (t_0, t) we assume the existence of a maximum of $|\bar{X}|$ and of $|\delta\Upsilon|$ and we call these $|\bar{X}|_M$ and $|\delta\Upsilon|_M$ respectively. We also assume the integral $\int_{t_0}^{t} |g(t')|dt'$ can be bounded by a number M. If

$$\int_{t_0}^{t} |g(t')|dt' < M, \tag{4.28}$$

then it follows that also

$$\int_{t'}^{t} |g(t'')|dt'' < M. \tag{4.29}$$

We thus find

$$|\delta X(t)| < |\bar{X}|_M \left(M|\delta\Upsilon|_M + M^2|\delta\Upsilon|_M^2 \exp\left(M|\delta\Upsilon|_M\right)\right). \tag{4.30}$$

The second term on the right-hand side is negligible compared to the first one for sufficiently small $|\delta\Upsilon|$.

4.2 Solutions away from the singularity

In regions I and III we will take

$$\Upsilon = n^2 + k/t^2, \qquad \delta\Upsilon = \frac{1}{\epsilon^2} O\left(\frac{\epsilon^b}{t^b}\right), \tag{4.31}$$

with b defined in (1.5). The solutions to the unperturbed differential equation (4.7) are given by

$$\sqrt{|t|}J_\alpha(|nt|), \qquad \sqrt{|t|}J_{-\alpha}(|nt|), \qquad \alpha = a - \frac{1}{2}, \tag{4.32}$$

where the Bessel functions, $J_\alpha(x)$ and $J_{-\alpha}(x)$, satisfy the differential equation

$$x^2\frac{\partial^2}{\partial x^2}J_\alpha(x) + x\frac{\partial}{\partial x}J_\alpha + \left(x^2 - \alpha^2\right) J_\alpha(x) = 0. \tag{4.33}$$

(This Bessel-negative-order-Bessel basis is more convenient for our purposes than the often-used Bessel-Neumann basis, as it approaches $|t|^a$ and $|t|^{1-a}$ for small values of t without mixing the two powers.)

The unperturbed "compression factor" in region I is then

$$\bar{\mathcal{C}}_I(t_1, t) = \sqrt{|t_1|}\sqrt{|t|}\frac{J_\alpha(-nt_1)J_{-\alpha}(-nt) - J_\alpha(-nt)J_{-\alpha}(-nt_1)}{W[\sqrt{|t|}J_\alpha(-nt), \sqrt{|t|}J_{-\alpha}(-nt)]}. \tag{4.34}$$

Using the series expansion of the Bessel function for small arguments (they will be evaluated at $t = -t_\epsilon$),

$$J_\alpha(x) \sim \left(\frac{x}{2}\right)^\alpha \frac{1}{\Gamma(\alpha+1)}, \qquad \alpha \neq -1, -2, -3, \ldots. \tag{4.35}$$

we can estimate the various contributions to (4.30), thereby constraining the correction to the unperturbed "compression factor". One can distinguish three cases:

(1) $a > 1, J_\alpha(-nt_1) \neq 0$, which yields

$$|\bar{\mathcal{C}}(t_1, t_\epsilon)| \propto \epsilon^{(1-c)(1-a)}, \quad |\bar{\mathcal{C}}|_M \propto \epsilon^{(1-c)(1-a)}, \quad |\delta\Upsilon|_M \propto \epsilon^{bc-2}, \quad M \propto \epsilon^{(1-c)(1-a)}. \tag{4.36}$$

From (4.30), $\delta\mathcal{C}(t_1,t_\epsilon)$ is negligible compared to $\mathcal{C}(t_1,t_\epsilon)$ if

$$c > \frac{a+1}{a+b-1}. \tag{4.37}$$

(2) $a<1$, $J_\alpha(-nt_1)\neq 0$, which yields

$$|\bar{\mathcal{C}}(t_1,t_\epsilon)| \propto \epsilon^{(1-c)(1-a)}, \qquad |\bar{\mathcal{C}}|_M \propto \epsilon^0, \qquad |\delta\Upsilon|_M \propto \epsilon^{bc-2}, \qquad M\propto\epsilon^0. \tag{4.38}$$

From (4.30), $\delta\mathcal{C}(t_1,t_\epsilon)$ is negligible compared to $\mathcal{C}(t_1,t_\epsilon)$ if

$$c > \frac{3-a}{b+1-a}. \tag{4.39}$$

(3) $J_\alpha(-nt_1)=0$, which yields

$$|\bar{\mathcal{C}}(t_1,t_\epsilon)| \propto \epsilon^{(1-c)a}, \qquad |\bar{\mathcal{C}}|_M \propto \epsilon^0, \qquad |\delta\Upsilon|_M \propto \epsilon^{bc-2}, \qquad M\propto\epsilon^0. \tag{4.40}$$

From (4.30), $\delta\mathcal{C}(t_1,t_\epsilon)$ is negligible compared to $\mathcal{C}(t_1,t_\epsilon)$ if

$$c > \frac{2+a}{b+a}. \tag{4.41}$$

Whichever of the three cases is realized, it suffices for c to be greater than a number less than 1, in order for the corrections to the unperturbed "compression factor" to be negligible for small values of ϵ. The discussion of interval III is completely parallel to what we have just presented.

4.3 Solutions in the near-singular region

The "unperturbed" equation in region II,

$$\frac{\partial^2}{\partial t^2}\bar{X}(t) + \frac{\lambda}{\epsilon^2}\Omega(t/\epsilon)\bar{X}(t) = 0, \tag{4.42}$$

is precisely that of the string center-of-mass motion. In order to simplify derivations, we shall assume $a_+ = a_-$, as required for well-defined zero-mode propagation (see section 3). The unperturbed "compression factor" in region II takes the form[2]

$$\bar{\mathcal{C}}_{\mathrm{II}}(t_i,t_f) = \frac{Q_{22}(\lambda)|t_i|^a t_f^{1-a} - Q_{11}(\lambda)|t_i|^{1-a}t_f^a - Q_{12}(\lambda)|t_i|^{1-a}t_f^{1-a}\epsilon^{2a-1} + Q_{21}(\lambda)|t_i|^a t_f^a \epsilon^{1-2a}}{2a-1}, \tag{4.43}$$

where the 2×2 matrix Q is defined by (3.12), and we have used $W[|t|^a,|t|^{1-a}] = 2a-1$. To study the perturbation we will first perform the scaling transformation $\eta = t/\epsilon$, $Y(\eta) = X(\eta\epsilon)$, which yields

$$\frac{\partial^2}{\partial\eta^2}Y(\eta) + \left(\epsilon^2 n^2 + \lambda\Omega(\eta)\right)Y(\eta) = 0. \tag{4.44}$$

[2] This formula follows from (3.4), (3.5) and (3.7) via (2.19). It is also the same as (44) of [9]. The expression given corresponds to small values of ϵ. The corrections are suppressed by powers of ϵ and do not contribute to the singular limit.

We now take

$$\Upsilon = \lambda\Omega(\eta), \qquad \delta\Upsilon = \epsilon^2 n^2. \tag{4.45}$$

If we now choose $f \sim \eta^a$, $g \sim \eta^{1-a}$ in (4.25), M of (4.28) for the region $(-t_\epsilon/\epsilon, t_\epsilon/\epsilon)$ (whose size, in η, is proportional to ϵ^{-c}) becomes (with the three factors coming from f, h and the size of the integration region):

$$M \propto \epsilon^{-ac}\epsilon^{-(1-a)c}\epsilon^{-c} = \epsilon^{-2c}. \tag{4.46}$$

Because there are only power laws involved in (4.43), the maximal value $\bar{\mathcal{C}}_M$ is of the same order as $|\bar{\mathcal{C}}(-t_\epsilon, t_\epsilon)|$. Furthermore, $|\delta\Upsilon|_M \propto \epsilon^2$ by construction. It then follows from (4.30) that

$$|\delta\mathcal{C}_{\rm II}| < \left(O\left(\epsilon^{2-2c}\right) + O\left(\epsilon^{4-4c}\exp\left(\epsilon^{2-2c}\right)\right)\right)|\bar{\mathcal{C}}_{\rm II}|. \tag{4.47}$$

The correction is negligible for any $c < 1$.

A subtlety in our above derivation deserves a comment (we would like to thank the JHEP referee for raising this point): one might have thought that the factor of n^2 in $\delta\Upsilon$ of (4.45) competes with the smallness of ϵ and undermines the validity of our considerations (for sufficiently large mode numbers). It is indeed true that, for each value of ϵ (each fixed resolved space), our analysis is only valid for modes with sufficiently small mode numbers (though this range of validity increases infinitely as ϵ is taken to 0). However, since the modes are completely independent, the limit for the motion of the entire string (if it exists) is exactly the same as if it were computed mode-by-mode. For that reason, n can be kept fixed in the derivations of this section, and the problem of n^2 competing with the smallness of ϵ does not arise. (This attitude guarantees reproducing the $\epsilon \to 0$ limit correctly for the entire set of modes, though it does not allow to draw conclusions on the uniformity of this limit with respect to n.)

4.4 Effective matching conditions

Having analyzed the "compression factors" on subintervals I, II and III, we can combine them into the total "compression factor" by applying (4.5). As has been shown above, there exist a number c in (4.5) between 0 and 1, such that the "compression factors" on subintervals I, II and III can be well approximated by the simplified expressions (4.34) and (4.43), with corrections suppressed by positive powers of ϵ. One can then substitute (4.34) and (4.43) into the right-hand-side of (4.5).

For $a > 1$ ($\lambda k < 0$), the Bessel functions featured in (4.34) blow up near the origin (the inverted harmonic oscillator is propelled off to infinity). This threatens the existence of an $\varepsilon \to 0$ limit. In appendix A, we display the divergences arising for $a > 3/2$. (For $1 < a < 3/2$, the limit may exist for individual string modes, but a consideration along the lines of section 5 would still indicate no consistent propagation for the entire string.) In any case, we shall not explore this case further since, as will be explained in section 6, free strings are not a good approximation to motion in such plane waves.

For $a < 1$ ($\lambda k > 0$), substituting (4.34) and (4.43) in (4.5) yields

$$\begin{aligned}\bar{\mathcal{C}}(t_1,t_2) =& \frac{\sqrt{-\pi t_1 t_2}}{2\sin\alpha\pi}\Big(Q_{22}(\lambda)J_{a-1/2}(-nt_1)J_{1/2-a}(nt_2) - Q_{11}(\lambda)J_{1/2-a}(-nt_1)J_{a-1/2}(nt_2)\\ &+ Q_{21}(\lambda)\epsilon^{1-2a}J_{a-1/2}(-nt_1)J_{a-1/2}(nt_2)\gamma_n\\ &- Q_{12}(\lambda)\epsilon^{2a-1}J_{1/2-a}(-nt_1)J_{1/2-a}(nt_2)\gamma_n^{-1}\Big), \quad t_1<0, \quad t_2>0,\end{aligned} \tag{4.48}$$

where γ_n are numbers originating from the coefficients of the power law expansion of the Bessel functions.

Note that the expression (4.48) has the same algebraic structure as the one derived for the center-of-mass motion in [9], except that $|t|^a$ and $|t|^{1-a}$ are replaced by $\sqrt{|t|}J_\alpha(|t|)$ and $\sqrt{|t|}J_{-\alpha}(|t|)$. Demanding that the $\epsilon \to 0$ limit should exist results in the condition

$$Q_{21}(\lambda) = 0. \tag{4.49}$$

It is exactly the same condition as the one for the existence of a singular limit of the center-of-mass motion (generically leading to a discrete spectrum for λ). Under the assumption of (4.49) we obtain in the singular limit

$$\begin{aligned}\mathcal{C}(t_1,t_2) = \sqrt{-t_1t_2}\,&\frac{Q_{22}(\lambda)J_{a-1/2}(-nt_1)J_{1/2-a}(nt_2) - Q_{11}(\lambda)J_{1/2-a}(-nt_1)J_{a-1/2}(nt_2)}{W[\sqrt{-t_1}J_{a-1/2}(-nt_1),\sqrt{-t_1}J_{1/2-a}(-nt_1)]},\\ t_1<0,& \quad t_2>0.\end{aligned} \tag{4.50}$$

The matching conditions across the singularity can now be derived rigorously by constructing two independent solutions to (4.1). Note that all the information necessary for such construction is encoded (cf. (4.3)) in the "compression factor" given by (4.50). A convenient shortcut for this procedure is to recall the representation (2.19) of $\mathcal{C}(t_1,t_2)$ in terms of two arbitrary independent solutions $f(t)$ and $h(t)$, and to read off the corresponding singular limit of the two solutions directly from (4.50). Writing $Q_{11}(\lambda) = q$ and $Q_{22}(\lambda) = -1/q$, we obtain as a basis of solutions,

$$\begin{aligned} Y_1(t) &= \sqrt{-t}J_{a-1/2}(-nt), & Y_2(t) &= \sqrt{-t}J_{1/2-a}(-nt), & t&<0,\\ Y_1(t) &= q\sqrt{t}J_{a-1/2}(nt), & Y_2(t) &= -\frac{\sqrt{t}}{q}J_{1/2-a}(nt), & t&>0.\end{aligned} \tag{4.51}$$

5 The singular limit for the entire string

As we have seen in the previous section, for $k\lambda > 0$, consistent propagation of the string center-of-mass across the singularity guarantees that all excited string modes also propagate in a consistent fashion. This is not sufficient, however, to define a consistent evolution for the whole string, since even small excitations of higher string modes can sum up to yield an infinite total energy [1]. As we shall see below, the condition of finite total string energy (after the singularity crossing) turns out to be very restrictive.

The total string excitation energy can be conveniently expressed in terms of the Bogoliubov coefficients for the higher string modes. To compute the latter, we shall form two different bases of solutions from (4.51) corresponding to purely positive and negative frequencies at large negative and large positive times. More specifically, using the asymptotic expansion for the Bessel functions

$$J_{\pm\alpha}(x) \sim \sqrt{\frac{2}{\pi x}}\cos\left(x \mp \alpha\frac{\pi}{2} - \frac{\pi}{4}\right), \qquad x \to \infty, \tag{5.1}$$

we construct

$$\begin{bmatrix}\phi_1^-\\ \phi_2^-\end{bmatrix} = \frac{i}{\sin(\alpha\pi)}\begin{bmatrix}-\exp(i\alpha\pi/2 - i\pi/4) & \exp(-i\alpha\pi/2 - i\pi/4)\\ \exp(-i\alpha\pi/2 + i\pi/4) & -\exp(i\alpha\pi/2 + i\pi/4)\end{bmatrix}\begin{bmatrix}Y_1(t)\\ Y_2(t)\end{bmatrix}, \tag{5.2}$$

such that,

$$\phi_1^-(t) \sim \sqrt{\frac{2}{\pi n}}\exp(int), \qquad \phi_2^-(t) \sim \sqrt{\frac{2}{\pi n}}\exp(-int), \qquad t \to -\infty. \tag{5.3}$$

Analogously, we introduce

$$\begin{bmatrix}\phi_1^+\\ \phi_2^+\end{bmatrix} = \frac{i}{q\sin(\alpha\pi)}\begin{bmatrix}\exp(-i\alpha\pi/2 + i\pi/4) & q^2\exp(i\alpha\pi/2 + i\pi/4)\\ -\exp(i\alpha\pi/2 - i\pi/4) & -q^2\exp(-i\alpha\pi/2 - i\pi/4)\end{bmatrix}\begin{bmatrix}Y_1(t)\\ Y_2(t)\end{bmatrix}, \tag{5.4}$$

such that

$$\phi_1^+(t) \sim \sqrt{\frac{2}{\pi n}}\exp(int), \qquad \phi_2^+(t) \sim \sqrt{\frac{2}{\pi n}}\exp(-int), \qquad t \to +\infty. \tag{5.5}$$

The two bases are related by a matrix made of Bogoliubov coefficients α_n and β_n:

$$\begin{bmatrix}\phi_1^+\\ \phi_2^+\end{bmatrix} = \begin{bmatrix}\alpha_n & \beta_n\\ \beta_n^* & \alpha_n^*\end{bmatrix}\begin{bmatrix}\phi_1^-\\ \phi_2^-\end{bmatrix} \tag{5.6}$$

For the Bogoliubov coefficients, we obtain the following expressions, independent of n:

$$\alpha_n = -\frac{1+q^2}{2q\sin(\alpha\pi)}, \tag{5.7}$$

$$\beta_n = i\frac{\exp(-i\pi\alpha) + q^2\exp(i\pi\alpha)}{2q\sin(\alpha\pi)}. \tag{5.8}$$

Here, $\alpha = \sqrt{1-4k\lambda}/2$. The total mass of the string after crossing the singularity is given by [1]

$$M = \sum_n n|\beta_n|^2. \tag{5.9}$$

Since the β_n are n-independent, M can only be finite[3] if $\beta_n = 0$ for all n. For $k\lambda > 0$, this cannot be achieved, since $0 < \alpha < 1/2$ and q is real. (For $k = 0$, which is the case of the "lightlike reflector plane" of [11], all β_n will vanish if $q^2 = 1$, which is satisfied automatically for any reflection-symmetric $\Omega(\lambda)$.)

[3]In general, one needs the uniformity of the $\epsilon \to 0$ limit of β_n with respect to n in order to analyze infinite sums as in (5.9). As remarked at the end of section 4.3, our considerations allow to draw immediate conclusions on the existence of the limit, but not on its uniformity. However, since M is a sum of positive numbers, it is obvious that it will diverge when the β_n approach an n-independent non-zero value (in the $\epsilon \to 0$ limit), irrespectively of whether this approach is uniform in n. For that reason, no further considerations are needed to draw our conclusions.

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6 Discussion

Before we recapitulate our main results, it shall be appropriate to make two observations.

First, one can ask what kind of cosmological singularities gives rise, when the Penrose limit is taken, to the plane wave singularities we have been considering. According to [7], if one starts with isotropic homogeneous cosmology of the type

$$ds^2 = -dt^2 + t^{2h} dx^i dx^i, \tag{6.1}$$

and performs a Penrose limit, one obtains a plane wave of the form (1.1)–(1.2) with

$$k\lambda = \frac{h}{(1+h)^2}. \tag{6.2}$$

Thus, positive values of $k\lambda$ correspond to positive h, i.e., Friedmann-like Big Bang singularities, and negative values of $k\lambda$ correspond to negative h, i.e., an infinite-expansion rather than an infinite-contraction singularity ("Big Rip").

Second, the dilaton field (discussed in more detail in appendix B) in the backgrounds of the type (1.1)–(1.2) takes the form [2]

$$\phi = \phi_0 + cX^+ + \frac{dk\lambda}{2} \ln X^+ \tag{6.3}$$

If $k\lambda$ is negative, this expression blows up near X^+ (and so does the string coupling) posing a serious threat to the validity of perturbative string theory, and of free string propagation as zeroth order approximation thereto.

For this reason of limited validity of the free string approximation when $k\lambda < 0$, we have paid relatively little attention to this case. What we could see is that, generically, it is hard to make excited string modes propagate consistently across the singularity (though it may still be possible to arrange such propagation by means of a judicious choice of the resolved profile $\Omega(\eta)$ of the plane wave). The issue, however, cannot be competently addressed within perturbative string theory on account of string coupling blow-up. Our considerations can be seen as a motivation to study these backgrounds in the context of non-perturbative matrix theory descriptions of quantum gravity (the Matrix Big Bang case of [4] corresponds to the 11-dimensional analog of the plane waves we have been considering compactified on a light-like circle with k taken to $-\infty$ [5].). Some steps in this direction have been taken in [5]. (Alternatively, one could try to construct plane wave backgrounds of the type (1.1)–(1.2) where the curvature of the metric is compensated by non-zero p-forms, rather than the dilaton, thus avoiding the dilaton blow-up problem.)

For the case of positive $k\lambda$, i.e., those plane waves that arise as Penrose limits of Friedmann-like cosmologies, it turns out that individual excited string modes propagate consistently across the singularity, whenever the center-of-mass of the string does. In those cases, the dilaton (6.3) is actually very large and *negative* near the singularity, and one can expect that free strings are a good approximation as far as propagation across the singularity is concerned (the string coupling is small in the near-singular region). However, for free strings, we find it impossible to maintain a finite total string energy after the

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singularity crossing, provided that the (scale-invariant) singularity is resolved in a way that does not introduce new dimensionful parameters. The only way out appears to be to allow hidden scales buried at the singular locus[4] (even though the space-time away from the singularity is scale-invariant). To contemplate the possible physical origins of such dimensionful scales is an interesting pursuit, outside the scope of the present publication.

Another relevant consideration would be the propagation of strings across plane wave singularities stronger than $1/(X^+)^2$. Unfortunately, at present, little can be said about this case, even for the center-of-mass motion.

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A Divergences for the case of the inverted harmonic oscillator

As remarked in section 4.4, for the case of $k\lambda < 0$ (inverted harmonic oscillator), divergences may arise in the evolution of excited string modes. These divergences may be seen via a blunt application of (4.5), but it will be more instructive to make their algebraic structure more explicit.

To this end, we shall derive a slightly different representation for the total "compression factor" in place of (4.5). One can start by rewriting (2.19) as

$$\mathcal{C}(t_1,t_2) = \frac{1}{W[f,h]}\begin{pmatrix} f(t_1) & h(t_1)\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} f(t_2) \\ h(t_2)\end{pmatrix} \tag{A.1}$$

For any two sets of solutions $\{f,h\}$ and $\{F,H\}$, the following relation holds:

$$\begin{pmatrix} f(t) \\ h(t)\end{pmatrix} = \frac{1}{W[F,H]}\begin{pmatrix} W[f,H] & -W[f,F] \\ W[h,H] & -W[h,F]\end{pmatrix}\begin{pmatrix} F(t) \\ H(t)\end{pmatrix}. \tag{A.2}$$

One can then take four sets of solutions: one approximated by $\{\sqrt{-t}J_\alpha(-nt), \sqrt{-t}J_{-\alpha}(-nt)\}$ in region I, two approximated by $\{Y_{1-}(t/\epsilon), Y_{2-}(t/\epsilon)\}$ and $\{Y_{1+}(t/\epsilon), Y_{2+}(t/\epsilon)\}$ in region II, and one approximated by $\{\sqrt{t}J_\alpha(nt), \sqrt{t}J_{-\alpha}(nt)\}$ in region III. One can then start with (A.1) written with the first of these four sets of solutions. In this representation, the functions featured in (A.1) are easily evaluated at

[4]If arbitrary resolutions, more general than (1.4), are allowed, for a given string mode, one should be able to reproduce (virtually) any matching conditions. This can be seen by assuming a particular form of solutions to the harmonic oscillator equation describing string propagation, and then reconstructing the plane wave profile necessary to produce this assumed motion. However, it is non-trivial to fit matching conditions for the entire tower of string modes in a particular geometrical resolution. For example, it is not obvious whether the matching conditions postulated in [2] should have any geometrical interpretation at all.

$t_1 < 0$, but not at $t_2 > 0$. One then consecutively applies (A.2), (3.7) and (A.2) again to insert the remaining three sets of solutions, with the Wronskians in (A.2) being evaluated at the boundaries of sub-regions. In the resulting expression, all the four sets of solutions occur only with the values of the arguments for which we have convenient approximations to these solutions, and the total compression factor can be evaluated. As a matter of fact, this is simply another way to write (4.5).

The divergent contributions to the total "compression factor" can be identified with particular Wronskians emerging from (A.2), when one constructs the total "compression factor" with the procedure outlined in the previous paragraph. For example, at the boundary of regions I and II, the following Wronskian occurs:

$$W[\sqrt{-t}J_{-\alpha}(-nt), Y_{2-}(t/\epsilon)]\Big|_{t=-t_\epsilon} \tag{A.3}$$

The leading terms of both functions featured in the Wronskian are proportional to $|t|^{1-a}$, and therefore cancel by virtue of antisymmetry of the Wronskian. However, the sub-leading contributions have a different functional form and do not have to cancel. For example, for $a > 3/2$, one may consider the contribution from the first sub-leading power-law correction to the Bessel function, and the leading term in Y_{2-}. This term is proportional to

$$W[|t|^{3-a}, |t|^{1-a}] \sim t^{3-2a}, \tag{A.4}$$

and furthermore it is not accompanied by any powers of ϵ in the total expression for $\mathcal{C}(t_1, t_2)$. For that reason, evaluating this term at $t = -t_\epsilon$ and taking the $\epsilon \to 0$ limit will produce a divergence.

B Background consistency and the singular limit for the dilaton

As we have seen in the course of main exposition, consistent free string propagation turns out to impose extremely stringent constraints on the treatment of scale-invariant dilatonic plane wave backgrounds. For that reason, it was not crucial for our picture to explore further conditions arising from supergravity equations of motion imposed on the background. However, for methodological completeness, we shall present considerations for the singular limit of the dilaton field, and examine how this condition combines with propagation of individual string modes. These derivation will not have much bearing on the outcome of the analysis in the main text, but they may be useful for pursuing various modifications of our present set-up.

If a time-dependent dilaton is used to support the curvature of the metric (1.1)–(1.2) in the context of string theory, the condition for conformal invariance of the world-sheet theory is given by [2]

$$R_{\mu\nu} = -2D_\mu D_\nu \phi. \tag{B.1}$$

We shall impose this equation for all X^+ in the resolved plane wave profile, and then examine the singular limit of the solutions for the dilaton. This is in contrast to the approach in [2], where the background consistency conditions at the singular locus were not discussed. The condition for conformal invariance (B.1) leads to the equation

$$\ddot{\phi}(t) = -\frac{\lambda d}{2\epsilon^2}\Omega(t/\epsilon) \tag{B.2}$$

for the dilaton where d is the number of transverse dimensions X^i. We want to consider the limit $\epsilon \to 0$ of the solution ϕ to this equation. In order for this limit to exist, the regularization Ω will have to fulfill extra conditions. Since, in the singular limit, the space-time is regular away from $X^+ = 0$, we can construct a solution $\phi(t)$ to the left of the singularity and another solution $\phi(t)$ to the right. The requirements for the singular limit of ϕ to exist then reduce to demanding that the jumps in $\phi(t)$ and in its first derivative $\dot{\phi}(t)$ are finite:

$$\Delta\phi = \int_{t_1}^{t_2} \dot{\phi}(t)dt = \Big[t\dot{\phi}(t)\Big]_{t_1}^{t_2} - \int_{t_1}^{t_2} t\ddot{\phi}(t)dt \tag{B.3}$$

$$= \Big[t\dot{\phi}(t)\Big]_{t_1}^{t_2} + \frac{\lambda d}{2}\int_{t_1/\epsilon}^{t_2/\epsilon} \eta\Omega(\eta)d\eta \tag{B.4}$$

and

$$\Delta\dot{\phi} = -\int_{t_1}^{t_2} \frac{\lambda d}{2\epsilon^2}\Omega(t/\epsilon)dt = -\frac{\lambda d}{2\epsilon}\int_{t_1/\epsilon}^{t_2/\epsilon} \Omega(\eta)d\eta \tag{B.5}$$

Thus, $\Delta\dot{\phi}$ can only be finite if

$$\int_{-\infty}^{+\infty} \Omega(\eta)d\eta = 0. \tag{B.6}$$

If that is the case, the first term in (B.4) is automatically finite, and we are left to demand finiteness of the second term

$$\lim_{\epsilon\to 0}\int_{t_1/\epsilon}^{t_2/\epsilon} \eta\Omega(\eta)d\eta < \infty. \tag{B.7}$$

If Ω is even and satisfies (1.5), this second condition is automatically satisfied.

B.1 An explicit example

We would now like to show that it is possible to combine the finite dilaton condition (B.6) with consistent propagation of individual string modes. Given the considerations in the main text, this translates into finding $\Omega(\eta)$ such that (B.6) is satisfied and, in addition,

$$\frac{\partial^2}{\partial\eta^2}Y(\eta) + \lambda\Omega(\eta)Y(\eta) = 0 \tag{B.8}$$

has a solution approaching $Y(\eta) \propto \eta^{1-a}$ for $\eta \to \pm\infty$. We shall apply inverse reconstruction to $\Omega(\eta)$, assuming some shape of this solution and adjusting it so as to satisfy

$$\int_{-\infty}^{+\infty} \frac{Y''(\eta)}{Y(\eta)}\,d\eta = 0. \tag{B.9}$$

This “inverse reconstruction” technique is generally useful for contemplating qualitative properties of various plane wave profiles in relation to the singular limit.

B.1.1 No-go theorem for $Y(\eta)$ without zero crossings

In constructing an appropriate $Y(\eta)$, it is important to decide whether it should have zeros. If Y has no zeros, Y'/Y is regular everywhere, and we can rewrite (B.9) as:

$$\int_{-\infty}^{+\infty} \frac{Y''(\eta)}{Y(\eta)}\, d\eta = \left[\frac{Y'(\eta)}{Y(\eta)}\right]_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} \frac{Y'^2(\eta)}{Y^2(\eta)}\, d\eta. \tag{B.10}$$

We now use $Y(\eta) \propto \eta^{1-a}$ for $\eta \to \pm\infty$, yielding

$$\int_{-\infty}^{+\infty} \frac{Y''(\eta)}{Y(\eta)}\, d\eta = \int_{-\infty}^{+\infty} \frac{Y'^2(\eta)}{Y^2(\eta)}\, d\eta > 0. \tag{B.11}$$

Therefore, if Y has no zeros, it is impossible to construct an $\Omega(\eta)$ that integrates to zero. One must permit zeros (say $Y(\eta_i) = 0$), and it is necessary to have bending points ($Y''(\eta_i) = 0$) at the same locations due to (B.8). We will aim at constructing a symmetric Ω, assuming that Y is symmetric and restricting our analysis to $\eta > 0$, and we will look for Y that has only one zero for $\eta > 0$.

B.1.2 Piece-wise construction of solution

We will now prove that it is possible to construct an Ω that integrates to zero for a Y that has one zero-crossing. Ω can be made arbitrarily smooth but for the simplicity of the proof we will allow Ω to have discontinuities. The main idea is to split the contributions to the integral

$$\int_0^\infty \Omega(\eta) \mathrm{d}\eta, \tag{B.12}$$

into two parts, separated by $\eta = \eta_M$. The part

$$\int_{\eta_M}^\infty \Omega(\eta) \mathrm{d}\eta, \tag{B.13}$$

will be chosen to be always positive. Then we prove that the contribution

$$\int_0^{\eta_M} \Omega(\eta) \mathrm{d}\eta, \tag{B.14}$$

can be made equal to any negative number while keeping the $\eta > \eta_M$ region intact. Therefore the total sum (B.12) can always be taken zero by adjusting the $\eta < \eta_M$ contribution.

We rewrite equation (B.8) as

$$\Omega = -\frac{1}{\lambda}\frac{Y''}{Y}, \tag{B.15}$$

and we take a piecewise $Y(\eta)$ (with a continuous first derivative),

$$Y(\eta) = \begin{cases} Y_1(\eta) & -\eta_M < \eta < \eta_M \\ Y_2(\eta) & |\eta| > \eta_M. \end{cases} \tag{B.16}$$

The function Y_2 is fixed throughout our considerations, and we demand that it asymptotes to the subdominant solution for large η: $Y_2 \to \eta^{1-a}$ with $2a = 1 + \sqrt{1 - 4\lambda}$. As mentioned

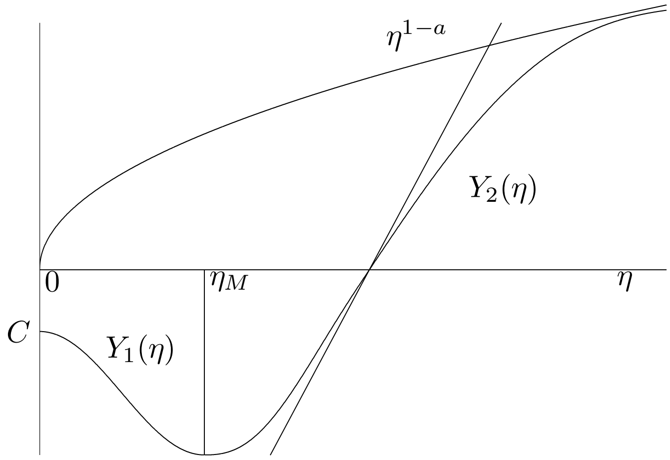

Figure 2. Piece-wise construction of $Y(\eta)$.

above, because of the denominator Y in Ω there needs to be a bending point for each crossing of the η-axis. Y_2''/Y_2 is *negative* everywhere at $\eta > \eta_M$. The splicing point η_M is taken to be a minimum, and we demand that $Y_1(\eta_M) = Y_2(\eta_M) \equiv Y(\eta_M)$. We take the following ansatz:

$$Y_1(\eta) = (C - Y(\eta_M)) \left(\frac{\eta^4}{\eta_M^4} - 2\frac{\eta^2}{\eta_M^2} \right) + C. \tag{B.17}$$

A pictorial representation of our assumed solution is given on figure 2. Due to the piecewise construction of Y it is clear that $\int \Omega(\eta) d\eta$ consists of a separate Y_1 and Y_2 contribution. The contribution of Y_2 (i.e., -$\int_{\eta_M}^{\infty} Y_2''/Y_2 d\eta$) will always be positive. It remains to be proven that Y_1 can contribute an arbitrarily negative value for fixed $Y(\eta_M)$ and η_M. With $\eta_M > 0$ and $\lambda > 0$, this is equivalent to asking that

$$\int_0^1 \frac{3y^2 - 1}{y^4 - 2y^2 + \frac{C}{C - Y(\eta_M)}} dy \tag{B.18}$$

can be set equal to an arbitrarily positive number. We know that $Y(\eta_M) \leq C < 0$, since Y_1 should not cross the η-axis and $\eta = \eta_M$ is a minimum. First, if $C = Y(\eta_M)$, the integral above is 0. Then, for $C \to 0^-$, with $\delta = -C/(C - Y(\eta_M)) > 0$, we find in the limit of $\delta \to 0$:

$$\int_0^1 \frac{3y^2 - 1}{y^4 - 2y^2 - \delta} dy \sim \frac{\pi}{2\sqrt{2\delta}}. \tag{B.19}$$

For $C \to 0^-$ or $\delta \to 0$ this becomes arbitrarily large and positive. As a consequence (B.14) can be made equal to any negative number (between 0 and $-\infty$), and (B.6) can be satisfied by appropriately adjusting $Y_1(\eta)$.

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D-brane potentials from multi-trace deformations in AdS/CFT

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Abstract: It is known that certain AdS boundary conditions allow smooth initial data to evolve into a big crunch. To study this type of cosmological singularity, one can use the dual quantum field theory, where the non-standard boundary conditions are reflected by the presence of a multi-trace potential unbounded below. For specific AdS_4 and AdS_5 models, we provide a D-brane (or M-brane) interpretation of the unbounded potential. Using probe brane computations, we show that the AdS boundary conditions of interest cause spherical branes to be pushed to the boundary of AdS in finite time, and that the corresponding potential agrees with the multi-trace deformation of the dual field theory. Systems with expanding spherical D3-branes are related to big crunch supergravity solutions by a phenomenon similar to geometric transition.

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1 Introduction

According to the AdS/CFT correspondence [1], type IIB string theory on global $AdS_5 \times S^5$ with N units of five-form flux,

$$\int_{S^5} \hat{G}_5 = N, \tag{1.1}$$

is dual to $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory on $\mathbb{R} \times S^3$ with gauge group SU(N). The string coupling g_s is related to the Yang-Mills coupling $g_{\rm YM}$ by $4\pi g_s = g_{\rm YM}^2$, while the radius of curvature $R_{\rm AdS}$ of both AdS_5 and S^5 is given in terms of the string length l_s by

$$\frac{R_{\rm AdS}^4}{l_s^4} = g_{\rm YM}^2 N. \tag{1.2}$$

The Poincaré patch of $AdS_5 \times S^5$ with N units of five-form flux appears as the near-horizon limit of N coincident D3-branes. It is dual to $\mathcal{N} = 4$ SYM theory on $\mathbb{R}^{1,3}$. This is an example of geometric transition [2, 3], where a spacetime with a number of branes (measured by the flux through a cycle surrounding the branes) is dual to a spacetime without branes but where the cycle has become non-contractible (the branes having been replaced by flux through the topologically non-trivial cycle). Since the D3-branes are BPS, they can be separated and placed at arbitrary positions in the dimensions transverse to their worldvolumes — such configurations correspond to the Coulomb branch of $\mathcal{N} = 4$ SYM theory on $\mathbb{R}^{1,3}$. D3-branes in $AdS_5 \times S^5$ act as domain walls, separating regions that differ by one unit of five-form flux (and thus have different radius of curvature). If one wanted to decrease the five-form flux by one unit, one could send one of the D3-branes to infinity; similarly, to increase N, one could send in parallel D3-branes from infinity. Near-horizon limits of certain Coulomb branch configurations of D3-branes have been shown to lead to specific deformations of $AdS_5 \times S^5$ [4].

$\mathcal{N} = 4$ SYM theory on $\mathbb{R} \times S^3$ does not have a Coulomb branch: it is lifted by the conformal coupling of the scalar fields to the curvature of the S^3, which effectively makes those scalars massive. This is consistent with the fact that global AdS cannot be obtained as the near-horizon limit of parallel D3-branes. However, D3-branes can still be used to increase the five-form flux N: a spherical D3-brane sent in from infinity is a domain wall; it will dynamically shrink and annihilate, leaving behind an AdS_5 with one more unit of five-form flux. The fact that a spherical probe D3-brane shrinks is not simply a consequence of it having a tension, since, in the regime of large radius, the leading effect of the tension (proportional to the fourth power of the radius of the D3-brane) is cancelled, because of a BPS relation between D3-brane tension and charge, by a compensating effect due to the four-form potential. Rather, it is a subleading quadratic potential that causes the shrinking [5, 6]. The counterpart in the dual gauge theory is the quadratic potential due to the conformal coupling of the transverse scalar fields.

With the standard supersymmetric boundary conditions, $AdS_5 \times S^5$ is stable. It is known, though, that a modification of the standard boundary conditions can introduce instabilities, allowing $AdS_5 \times S^5$ to tunnel into a cosmological spacetime with a big crunch singularity, i.e. a spacelike singularity reaching the boundary of AdS in finite global time [7]. In the dual gauge theory, the modification of the AdS boundary conditions corresponds to adding an unstable double trace potential to $\mathcal{N} = 4$ SYM, allowing operators to become infinite in finite time [8–11]. The aim of the present paper is to provide a D3-brane interpretation of this instability. We will show that with modified boundary conditions, a spherical probe D3-brane feels a negative quartic potential in addition to its positive quadratic potential. As a consequence, AdS can nucleate spherical D3-branes that are subsequently stretched to infinite size in finite time.[1] Every spherical D3-brane that is nucleated and stretched to infinity leaves behind a spacetime with one less unit of five-form flux, which is thus more strongly curved than the original AdS. In a gravity approximation (which in reality breaks down when the radius of curvature becomes of order the string scale), the result is a big crunch singularity.

Recently, a similar duality has been proposed between AdS_4 compactifications of M-theory or type IIA string theory and ABJM theory, an $\mathcal{N} = 6$ superconformal $\mathrm{U}(N) \times \mathrm{U}(N)$ Chern-Simons theory with opposite levels k and $-k$, respectively [13]. An unstable

[1] In the context of the AdS/CFT correspondence, it was shown in [12] that a closely related effect develops for non-supersymmetric spherical branes violating the BPS bound. In our present situation, the branes classically saturate the BPS bound and the repulsive force is generated by quantum corrections sensitive to the boundary conditions.

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triple trace deformation of ABJM theory was studied in [14]; as in [7], the corresponding boundary condition in the bulk AdS_4 allows smooth initial data to evolve into a big crunch singularity. We will show that this instability corresponds to spherical M2-branes or D2-branes being stretched to infinity in finite time, due to a potential generated by the modified boundary conditions.

In [8], a double trace deformation of $\mathcal{N} = 4$ SYM was used to obtain a dual field theory description of a big crunch singularity, and an attempt was made to use self-adjoint extensions to evolve the system beyond the big crunch. This model is now understood not to be under good computational control, though, and a new proposal in the context of ABJM theory will appear in [15]. Our present work shows that finding a consistent self-adjoint extension amounts to specifying what happens when spherical D-branes are stretched to infinite radius in finite time. While at present it is unclear what can be learned from this new perspective, we hope that it will turn out to be useful in addressing these hard questions. Simpler systems for which our results may be useful are the hairy black hole solutions of [16], which are dual to stable multi-trace deformations and should be related to spherical D-branes with finite radius.

This paper is organized as follows. In section 2, we study spherical probe D3-branes in global $AdS_5 \times S^5$ with modified boundary conditions. In section 3, we do the same for spherical probe M2-branes or D2-branes in global AdS_4 compactifications. In appendix A, we compute brane potentials in Poincaré coordinates, which supplement the computations in global coordinates in the main text.

2 D3-branes in $AdS_5 \times S^5$

In this section, we study spherical D3-branes in global $AdS_5 \times S^5$. This theory allows a consistent truncation to five-dimensional gravity with a negative cosmological constant coupled to a single scalar field [17]. This scalar field corresponds to quadrupole deformations of S^5 and saturates the Breitenlohner-Freedman bound [18]. We will compute the D-brane effective potential as a function of the boundary condition on this bulk scalar field. Specifically, we will focus on boundary conditions corresponding to a classically marginal double trace deformation of $\mathcal{N} = 4$ SYM theory.

In the Feynman diagrams of interest, the D-brane emits a virtual scalar particle, which then interacts with the boundary before being reabsorbed by the D-brane. To compute such diagrams, we need two main ingredients: the coupling of the D3-branes to the bulk scalar field, and the effect of the boundary condition on the bulk scalar field. The coupling can be obtained from the well-known D-brane action and from the consistent truncation ansatz expressing the ten-dimensional bulk fields in terms of the five-dimensional metric and scalar field. The effect of the boundary conditions can be computed in two different ways. On the one hand, a modification of the boundary condition corresponds to adding a boundary term to the bulk action, which gives rise to a (quadratic) vertex that should be included in Feynman diagrams. The advantage of this approach is that it extends to non-linear boundary conditions (such as the ones we will study in section 3). On the other hand, the (linear) boundary conditions we are focusing on in this section can be fully taken into account by using a modified propagator for the scalar field. This approach has the advantage that it effectively resums diagrams with arbitrarily many boundary vertices inserted.

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From the point of view of the full string theory on $AdS_5 \times S^5$, one might wonder whether it is sufficient to compute the effective potential in the truncated five-dimensional theory. In particular, spherical D3-branes can also emit and reabsorb many other fields, which are not described by the consistent truncation. The point is, however, that only the bulk scalar field of the consistent truncation is directly affected by the modified boundary conditions: contributions from emission and reabsorption of other fields are the same as in the standard supersymmetric theory. For our purposes, it is therefore justified to work within the framework of the consistent truncation.

In section 2.1, we review the basic setup, in particular the relation between modified boundary conditions in AdS and unstable double trace deformations in SYM. In section 2.2, we use the consistent truncation ansatz to determine the couplings of a spherical D3-brane to the bulk scalar field of interest. In section 2.3, we compute the propagator of the bulk scalar field, for standard as well as modified boundary conditions. In section 2.4, we compute the D-brane effective potential in the two ways described above. In section 2.5, we use the Coulomb branch solutions of [4] to provide additional evidence that the big crunch singularity in the supergravity solutions of [8] is due to branes being pushed to the conformal boundary of AdS.

2.1 Setup

Type IIB supergravity compactified on S^5 can be consistently truncated to five-dimensional gravity coupled to a single SO(5) invariant scalar φ [17]. From the ten-dimensional point of view, φ arises from an SO(5) invariant quadrupole distortion of S^5. The bulk action reads

$$S = \frac{V_{S^5}}{\kappa_{10}^2}\int d^5x\sqrt{-g}\left[\frac{R}{2} - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{4R_{\text{AdS}}^2}\left(15e^{2\gamma\varphi} + 10e^{-4\gamma\varphi} - e^{-10\gamma\varphi}\right)\right], \qquad (2.1)$$

where $\gamma = \sqrt{2/15}$, $2\kappa_{10}^2 = (2\pi)^7\alpha'^4 g_s^2$, and $V_{S^5} = \pi^3 R_{\text{AdS}}^5$ is the volume of the internal manifold. The potential reaches a negative local maximum when the scalar vanishes; this is the maximally supersymmetric AdS_5 state, corresponding to the unperturbed S^5 in the type IIB theory. At linear order around the AdS solution, the scalar obeys the free wave equation with a mass that saturates the Breitenlohner-Freedman [18] bound[2] $m^2 = -4/R_{\text{AdS}}^2$. With the usual supersymmetric boundary conditions, AdS_5 is stable.

In global coordinates, the AdS_5 metric takes the form

$$ds^2 = -\left(1 + \frac{r^2}{R_{\text{AdS}}^2}\right)dt^2 + \frac{dr^2}{1 + \frac{r^2}{R_{\text{AdS}}^2}} + r^2 d\Omega_3^2\,. \qquad (2.2)$$

In all asymptotically AdS solutions, the scalar φ decays at large radius as

$$\varphi(x,r) = \frac{\alpha(x)\ln(r\mu)}{r^2} + \frac{\beta(x)}{r^2}\,. \qquad (2.3)$$

Throughout this section, we denote the five-dimensional bulk coordinates by (r,x), where x collectively denotes the time and three angular coordinates. The arbitrary scale μ,

[2] In $d+1$ dimensions, $m_{\text{BF}}^2 = -d^2/(4R_{\text{AdS}}^2)$.

necessary to define the logarithm, will be chosen to be $\mu = 1/R_{\rm AdS}$ for most of this section (we will comment on this after (2.40)). The standard boundary conditions on the scalar field would set $\alpha = 0$. This choice preserves the full AdS symmetry group and has empty AdS as its stable ground state. However, in the mass range $m^2_{\rm BF} \le m^2 < m^2_{\rm BF} + 1/R^2_{\rm AdS}$, one can consider more general boundary conditions of the form [10]

$$\alpha = -\frac{\delta W}{\delta \beta}\,, \tag{2.4}$$

where $W(\beta)$ is an arbitrary real smooth function.

We will be interested in scalar field boundary conditions

$$\alpha(x) = f\beta(x), \tag{2.5}$$

where f is an arbitrary constant (for $f > 0$, smooth initial data can develop a big crunch singularity [8]). This boundary condition does not preserve supersymmetry and breaks the asymptotic AdS symmetries to $\mathbb{R} \times \mathrm{SO}(4)$ [19].

To obtain the boundary condition (2.5) from a variational principle, one adds boundary terms to the bulk action (2.1). To do this in a precise way, we provide an IR regulator in the bulk by restricting the radial coordinate to $0 \le r \le \Lambda$. (Through the UV/IR correspondence, the location Λ of the regularized boundary in the bulk will correspond to a UV cutoff in the dual field theory). The boundary condition $\alpha = f\beta$ is obtained from the boundary term in the variation of the action if we add to the scalar field action (2.1) the term

$$S_{\rm bdy} = \frac{V_{S^5}}{\kappa^2_{10} R_{\rm AdS}} \int_\partial d^4x \sqrt{g_{\rm bdy}} \left[-1 + \frac{f}{2\left(1 + f\ln\left(\Lambda/R_{\rm AdS}\right)\right)}\right] \varphi^2. \tag{2.6}$$

The first term also appeared in [20] (see also [21]).

The AdS/CFT correspondence states that type IIB string theory on global $AdS_5 \times S^5$ with standard ($f = 0$) boundary conditions is dual to $\mathcal{N} = 4$ SYM theory on $\mathbb{R} \times S^3$. Because of the conformal coupling to the curvature of S^3, which we choose to be of unit radius, the six adjoint scalar fields Φ_j of $\mathcal{N} = 4$ SYM effectively get masses $m^2 = 1$.[3] According to [10, 11, 20], changing the boundary condition on φ to (2.5) with non-zero f corresponds to adding a double trace potential to the SYM action,

$$S = S_0 + \frac{f}{2}\int \mathcal{O}^2, \tag{2.7}$$

where $\mathcal{O}$ is the dimension 2 chiral primary operator dual to φ,

$$\mathcal{O} = c\,\mathrm{Tr}\left[\Phi_1^2 - \frac{1}{5}\sum_{i=2}^{6}\Phi_i^2\right] \tag{2.8}$$

with c a normalization constant of order $1/N$ (in conventions such that the fields Φ_i have canonical kinetic terms). The coupling f in (2.7) is classically marginal but in fact marginally relevant for $f > 0$ [10].

[3] For a d-dimensional boundary, the mass would be $m^2 = \frac{d-2}{4(d-1)} R_{S^{d-1}}$, where $R_{S^{d-1}}$ is the Ricci scalar of S^{d-1}.

The duality between type IIB string theory on (the Poincaré patch of) $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM theory (on $\mathbb{R}^4$) can be obtained by taking a decoupling limit of a system of coincident D3-branes [1]. In this picture, the eigenvalues of Φ_j correspond to D-brane positions, and the double trace potential in (2.7) provides a quartic potential for these D-brane positions. For $f > 0$, the potential is unbounded below and sufficiently strong to push eigenvalues to infinity in finite time. Global $AdS_5 \times S^5$, which is the background of interest in [7, 8], does not straightforwardly appear as a near-horizon limit of D3-branes. One could still expect that there should be a similar D-brane interpretation of the unstable potential in (2.7). It is natural to assume that spherical D3-branes will play a role in this, as in [5, 6]. The main purpose of the present paper is to make this picture precise by computing the effective potential felt by spherical probe D3-branes as a function of the boundary conditions.

2.2 Coupling of the bulk scalar field to spherical D3-branes

The fact that the action (2.1) is a consistent truncation of type IIB supergravity compactified on S^5 means that with any solution of (2.1) one can associate a solution of the full type IIB supergravity equations of motion. The lift to ten dimensions is explicitly given in [22]. Let $F = e^{\gamma\varphi}$ (with γ defined after (2.1)) and $\Delta = F\sin^2\xi + F^{-5}\cos^2\xi$, where ξ is a coordinate in terms of which the metric of the unit sphere would read

$$d\Omega_5^2 = d\xi^2 + \sin^2\xi d\Omega_4^2\,, \tag{2.9}$$

with $0 \leq \xi \leq \pi$. The full ten-dimensional metric is

$$ds_{10}^2 = \Delta^{1/2} ds_5^2 + R_{\rm AdS}^2\left[F^4\Delta^{1/2}d\xi^2 + F^{-1}\Delta^{-1/2}\sin^2\xi d\Omega_4^2\right]\,. \tag{2.10}$$

The self-dual five-form $\hat{G}_5 = G_5 + *G_5$ is determined by

$$G_5 = -\frac{U}{R_{\rm AdS}}\epsilon_5 + 6R_{\rm AdS}\sin\xi\cos\xi F^{-1} * dF\wedge d\xi\,, \tag{2.11}$$

where we have denoted

$$U = -3F^2\sin^2\xi + F^{-10}\cos^2\xi - F^{-4} - 4F^{-4}\cos^2\xi\,. \tag{2.12}$$

In (2.11), ϵ_5 and $*$ are the five-dimensionals volume-form and dual.

To compute the coupling of a spherical D3-brane to the scalar field φ, we consider a probe D3-brane in the ten-dimensional lifted solution. In our computation of the effective potential for the D3-brane radius, we will only need the source term for the bulk scalar,[4] so we work in a linearized approximation of the coupled scalar-gravity system about the AdS background. The action of the probe brane is

$$S_{D3} = S_{\rm DBI} + S_{\rm WZ} = -\tau_3\int d^4x\sqrt{-\hat{G}} + \mu_3\int\hat{C}_4\,, \tag{2.13}$$

[4]To compare with the deformation (2.7) of SYM, we will also need the kinetic terms for the scalar fields on the D3-brane world-volume.

where $\hat{G}$ is the determinant of the pull-back of the ten-dimensional metric to the D3-brane world-volume and $d\hat{C}_4 = \hat{G}_5$. The tension and charge are given by $\tau_3 = \mu_3 = \sqrt{\pi}/\kappa_{10}$. In the static gauge, the Dirac-Born-Infeld action includes the terms

$$S_{\rm DBI} = -\tau_3 \int d^4x \sqrt{-\hat{g}} \left[1 - 5\gamma\varphi \left(\cos^2\xi - \frac{1}{5}\sin^2\xi\right) + \frac{1}{2} g_{ij}\partial_a x^i \partial^a x^j\right], \tag{2.14}$$

where $\hat{g}$ is the determinant of the pull-back of the five-dimensional metric $g_{\mu\nu}$ to the four-dimensional world-volume, the index a labels the four coordinates along the D3-brane world-volume, the index i runs over the six transverse dimensions, and $\gamma = \sqrt{2/15}$ was introduced in (2.1). We rewrite the Wess-Zumino action as an integral over the five-dimensional volume enclosed by the D3-brane

$$S_{\rm WZ} = \mu_3 \int_{V_5} \hat{G}_5 = \frac{\mu_3}{R_{\rm AdS}} \int d^5x \sqrt{-g} \left[4 - 10\gamma\varphi \left(\cos^2\xi - \frac{1}{5}\sin^2\xi\right)\right], \tag{2.15}$$

where g denotes the determinant of the bulk metric. From (2.14) and (2.15), we read off the sources of the bulk scalar field.

We choose the bulk geometry to be AdS_5 in global coordinates (2.2), so that $\sqrt{-\hat{g}} = r^3[1 + r^2/R_{\rm AdS}^2]^{1/2}\sqrt{g_{S^3}}$ and $\sqrt{-g} = r^3\sqrt{g_{S^3}}$, and specialize to a spherical D3-brane of radius R in AdS_5 that is localized at a point on the S^5. By x we collectively denote the time coordinate and the coordinates on S^3. Due to the SO(5) symmetry of the problem, the location of the brane on S^5 will only enter the action through the coordinate ξ on S^5 (see (2.9)). We will be interested in D-branes near the conformal boundary of spacetime, in particular spherical branes with radius $R \gg R_{\rm AdS}$, for which the sources $\mathcal{J} \equiv \frac{1}{\sqrt{g}}\frac{\delta S}{\delta\varphi}\Big|_{\varphi=0}$ for φ reduce to

$$\mathcal{J}_{\rm DBI}(r) = 5\gamma\frac{\tau_3}{R_{\rm AdS}}\left(\cos^2\xi - \frac{1}{5}\sin^2\xi\right) r\delta(r-R), \tag{2.16}$$

$$\mathcal{J}_{\rm WZ}(r) = \begin{cases} -10\gamma\dfrac{\mu_3}{R_{\rm AdS}}\left(\cos^2\xi - \dfrac{1}{5}\sin^2\xi\right) & r \le R \\ 0 & r > R\,. \end{cases} \tag{2.17}$$

In fact, these expressions for the sources are valid not only in a pure AdS background, but also for branes near the boundary of more general asymptotically AdS backgrounds. Considering a static, spherically symmetric ansatz

$$ds_5^2 = -e^{-2\delta(r)} f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_3^2 \tag{2.18}$$

and solving the equations of motion following from (2.1),

$$\delta'(r) = -\frac{1}{3} r\varphi'(r)^2, \tag{2.19}$$

$$rf'(r) - 2 + 2f(r) = -\frac{1}{3}r^2\left[f(r)\varphi'(r)^2 + 2V(\varphi)\right], \tag{2.20}$$

$$f(r)\left[\varphi'(r) + r\varphi''(r)\right] = r\frac{\partial V}{\partial\varphi} + \frac{2}{3}\varphi'(r)\left(r^2V(\phi) - 3\right), \tag{2.21}$$

we find the asymptotic behavior

$$f(r) \sim 1 + \frac{r^2}{R_{\rm AdS}^2}\,, \tag{2.22}$$

$$\varphi(x,r) \sim \frac{1}{r^2}\left[\alpha(x)\ln(r\mu) + \beta(x)\right]\,, \tag{2.23}$$

$$\delta(x,r) \sim \frac{1}{3}\alpha(x)^2\frac{\ln^2(r\mu)}{r^4}\,. \tag{2.24}$$

In the limit of large radial coordinate we are interested in, (2.24) will not affect the computation of the D3-brane effective potential, since it will contribute to subleading order in $1/r$.

After computing the effective potential for the D3-brane transverse coordinates, we will want to compare it with the deformation (2.7) of the dual SYM theory. For that purpose, it will be useful to relate R and the S^5 angles to canonically normalized scalar fields. From (2.14), we can see that, for $R \gg R_{\rm AdS}$, the scalar fields

$$\phi_1 \equiv \sqrt{\tau_3}R_{\rm AdS}R\cos\xi\,, \quad \phi_2 \equiv \sqrt{\tau_3}R_{\rm AdS}R\,\sin\xi\cos\Omega_1\,, \quad \ldots \tag{2.25}$$

have canonical kinetic term

$$S_{\rm kin} = -\frac{1}{2}\int d^4\tilde{x}\,\partial_\alpha\phi_i\partial^\alpha\phi_i \tag{2.26}$$

in the coordinate system $\tilde{x}^\alpha = (\tilde{t} \equiv t/R_{\rm AdS}, \Omega_i)$ with metric

$$d\tilde{s}^2 = -d\tilde{t}^2 + d\Omega_3^2\,. \tag{2.27}$$

To make contact with $\mathcal{N}=4$ SYM, the fields ϕ_i of (2.25) play the role of eigenvalues of the fields Φ_i in (2.8):

$$\Phi_i = {\rm diag}(\phi_i, 0, \ldots, 0)\,, \qquad i = 1, \ldots, 6\,. \tag{2.28}$$

2.3 Propagator of the bulk scalar field

We now turn to the computation of the bulk propagator for the scalar field (satisfying the boundary condition (2.5)). To solve the scalar equation of motion following from (2.1), we separate variables writing

$$\varphi(x,r) = e^{-i\omega t}Y_{\ell,m}(\Omega)\Psi(r)\,, \tag{2.29}$$

where Y_ℓ (with $\ell \geq 0$) is the $\ell^{\rm th}$ spherical harmonics on S^3, satisfying

$$\nabla_{S^3}^2 Y_\ell = -\ell(\ell+2)Y_\ell\,. \tag{2.30}$$

Letting $a = 1 + \frac{1}{2}(\ell+\omega)$ and $b = 1 + \frac{1}{2}(\ell-\omega)$, and performing the change of coordinates

$$V = \frac{r^2}{R_{\rm AdS}^2 + r^2}\,, \tag{2.31}$$

the propagator is constructed from the following two radial solutions [23]. The first solution,

$$\Psi_1(V) = (1-V)V^{\ell/2}{}_2F_1(a, b, a+b; V)\,, \tag{2.32}$$

satisfies the regularity condition

$$S_{\text{origin}} = \lim_{r\to 0}\int_{r\,fixed} d^4x\sqrt{g}g^{rr}\varphi\partial_r\varphi \to 0 \tag{2.33}$$

at the origin. (For this solution, the boundary term of the classical action vanishes at the origin and therefore we do not have contributions to correlation functions of the dual field theory coming from the interior of the spacetime [23].) The second solution,

$$\begin{aligned}\Psi_2(V) = {} & (1-V)V^{\ell/2}\{{}_2F_1(a,b,1;1-V)\,[1+C_\infty\ln(1-V)] + \\ & C_\infty\sum_{k=1}^{\infty}(1-V)^k\frac{(a)_k(b)_k}{(k!)^2}[\psi(a+k)+\psi(b+k)-2\psi(1+k)-\psi(a)-\psi(b)-2\gamma_E]\},\end{aligned} \tag{2.34}$$

where γ_E denotes Euler's constant, satisfies the boundary conditions (2.5) defined with the scale $\mu = 1/R_{\text{AdS}}$, provided that we choose

$$C_\infty = -\frac{f}{2}. \tag{2.35}$$

Combining the two expressions with the appropriate normalization factor, we obtain the Feynman propagator

$$\begin{aligned}G_f(x,V;x',V') = {} & -\frac{\kappa_{10}^2}{R_{\text{AdS}}^2 V_{S^5}}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\sum_{\ell,m}\frac{\Gamma(a)\Gamma(b)}{\ell!(2\ell+2)}\frac{1}{1-C_\infty[\psi(a)+\psi(b)+2\gamma_E]}\times \\ & e^{-i\omega(t-t')}Y_{\ell,m}(\Omega)Y_{\ell,m}(\Omega')\big[\theta(V'-V)\Psi_1(V)\Psi_2(V')+\theta(V-V')(V\leftrightarrow V')\big].\end{aligned} \tag{2.36}$$

2.4 D3-brane effective potential

Consider a probe D3-brane extended along a three-sphere with radius R in global AdS_5 and localized at a point in S^5. In perturbation theory, the leading contribution to the D-brane effective potential is obtained by evaluating the D-brane action (2.13) in the $AdS_5\times S^5$ background. Combining (2.14) and (2.15) and making use of the BPS relationship $\tau_3 = \mu_3$, the leading order terms in the radial coordinate cancel among the two contributions. The term that survives in the DBI action results in an attractive quadratic potential $V \sim R^2$, corresponding to the conformal coupling of the massless scalar fields in the dual SYM theory on $\mathbb{R}\times S^3$ [5, 6]. This contribution is clearly independent of f, i.e., of the boundary condition on the bulk scalar field. (Note that $AdS_5 \times S^5$ is compatible with the boundary conditions (2.5) we consider. This was not the case for the boundary conditions studied in [6].)

The first contribution that is sensitive to the boundary condition is a diagram in which the brane emits and reabsorbes a φ particle. This diagram involves a φ propagator, which depends on f according to (2.36) with (2.35). Using this propagator and the sources (2.16), (2.17), we find the following term in the D3-brane effective action:

$$\begin{aligned}S_{\text{eff}} &= \frac{1}{2}\int d^4x\int_0^R dr\sqrt{g}\,[\mathcal{J}_{\text{BI}}(r)+\mathcal{J}_{\text{WZ}}(r)]\times \\ &\qquad \int d^4x'\int_0^R dr'\sqrt{g}\,G_f(x,r;x',r')\,[\mathcal{J}_{\text{BI}}(r')+\mathcal{J}_{\text{WZ}}(r')] \\ &= f\frac{5}{12}\frac{\tau_3^2\kappa_{10}^2}{V_{S^5}}\int d^4xR^4\left(\cos^2\xi-\frac{1}{5}\sin^2\xi\right)^2.\end{aligned} \tag{2.37}$$

Using the field redefinition (2.25) and the change of variables that brings the boundary metric in the form (2.27), we can rewrite the effective potential as

$$\int d^4\tilde{x}\, V_{\text{eff}}(\tilde{x}) = -f\frac{5\pi^2}{3N^2}\int d^4\tilde{x}\left[\phi_1^2 - \frac{1}{5}\sum_{i=2}^{6}\phi_i^2\right]^2 . \tag{2.38}$$

For $f > 0$, this is a quartic potential that pulls the spherical branes to the boundary of AdS. This potential agrees with what one would expect based on the dual $\mathcal{N} = 4$ SYM theory. In particular, it is consistent with the $\mathcal{O}^2$ structure of (2.7) with (2.8), as well as with the N-dependence of the deformation. In fact, the computation can be easily generalized to configurations with more than one brane.

This is the main result of this section, and we could stop here. However, our derivation crucially relied on the fact that the boundary condition (2.5) is linear, so that it could be fully taken into account by the modified propagator (2.36). Equivalently, the boundary term (2.6) is quadratic in φ, so that its effect can be absorbed in a modification of the propagator. This class of boundary conditions is very special. In fact, in the next section we will be interested in a non-linear boundary condition corresponding to a cubic boundary term and a triple trace interaction in the dual field theory. Therefore, we will now compute the D3-brane potential in a way that easily generalizes to non-linear boundary conditions. The idea is to work with the standard $f = 0$ propagator (corresponding to supersymmetric boundary conditions) and to treat the f-dependent boundary term in (2.6) as an interaction. Then, as illustrated in figure 1, a virtual φ particle emitted by a D3-brane can propagate to the boundary and "feel" the f-dependent boundary interaction before being reabsorbed by the D3-brane (the effect of the f-independent boundary term is already accounted for in the $f = 0$ propagator). In fact, the virtual φ particle can interact with the boundary an arbitrary number of times before being reabsorbed (see figure 2). Our previous computation, where the effect of the modified boundary condition was incorporated in a modified propagator, amounts to a resummation of all these contributions (which is possible for linear boundary conditions but not in more general cases). Let us thus compute the contribution to the D3-brane effective potential from a virtual φ particle interacting with the boundary a single time. Using the $f = 0$ propagator (2.36) (with $C_\infty = 0$) and the expressions (2.16), (2.17) for the sources, we find

$$\begin{aligned} S_{\text{eff}} = &\frac{f}{2\left(1 + f\ln\frac{\Lambda}{R_{\text{AdS}}}\right)}\frac{V_{S^5}}{R_{\text{AdS}}\kappa_{10}^2}\int_\partial d^4x\sqrt{g_{\text{bdy}}}\int d^5x'\sqrt{g}\left[\mathcal{J}_{\text{BI}}(r') + \mathcal{J}_{\text{WZ}}(r')\right]\times \\ &G_{f=0}(x', r'; x, \Lambda)\int d^5x''\sqrt{g}G_{f=0}(x, \Lambda; x'', r'')\left[\mathcal{J}_{\text{BI}}(r'') + \mathcal{J}_{\text{WZ}}(r'')\right] , \end{aligned} \tag{2.39}$$

which becomes

$$\int d^4\tilde{x}\, V_{\text{eff}}(\tilde{x}) = -\frac{f}{1 + f\ln\frac{\Lambda}{R_{\text{AdS}}}}\frac{5\pi^2}{3N^2}\int d^4\tilde{x}\left[\phi_1^2 - \frac{1}{5}\sum_{i=2}^{6}\phi_i^2\right]^2 . \tag{2.40}$$

The difference between (2.38) and (2.40) is that f got replaced with $f/[1 + f\ln(\Lambda/R_{\text{AdS}})]$ (which formally vanishes when the cutoff Λ is removed). From the point of view of our computations, the difference corresponds to the diagrams in figure 2, with multiple boundary

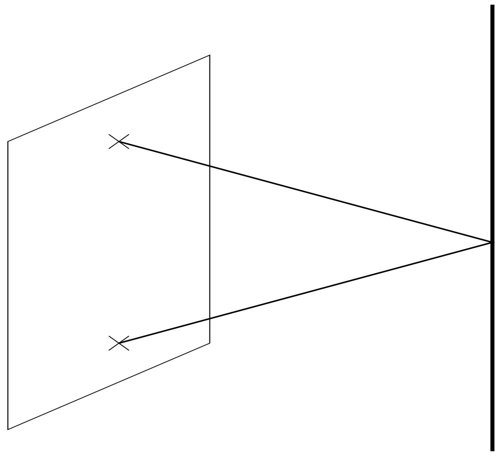

Figure 1. A virtual φ particle is emitted by the D3-brane, interacts with the boundary at $r = \Lambda$ and is reabsorbed by the brane.

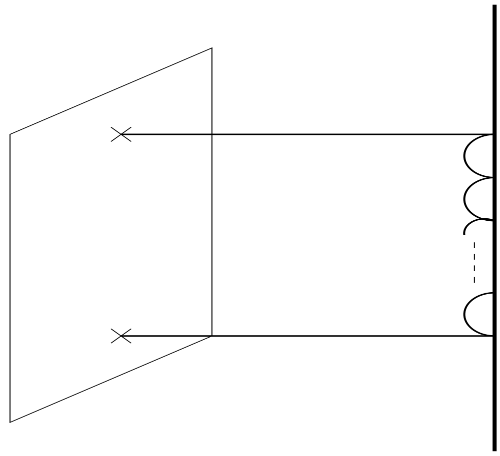

Figure 2. A virtual φ particle emitted by the D3-brane interacts an arbitrary number of times with the boundary before reabsorption.

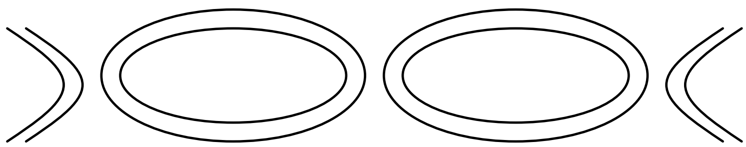

Figure 3. A two-loop example of factorizable diagrams that survive in the large N limit and renormalize the coupling f.

interactions — taking them into account will convert $f/[1 + f \ln(\Lambda/R_{\text{AdS}})]$ into f. From a dual field theory point of view, the difference lies in the scale at which the couplings are defined (cf. [10]). When expressing the asymptotic behavior (2.3) of the scalar field, we had to choose a scale μ to define the logarithm $\ln(r\mu)$, and we chose[5] $\mu = 1/R_{\text{AdS}}$, the scale appearing in the metric (2.2). This scale corresponds to a renormalization scale in the boundary theory [10]. On the other hand, $f/[1 + f \ln(\Lambda/R_{\text{AdS}})]$ is the coupling defined at the UV cutoff scale Λ/R_{AdS}^2 of the dual field theory. From a large N field theory perspective, the relation between the couplings at both scales is given by a resummation of (factorizable) planar diagrams with an arbitrary number of loops — the two-loop diagram is drawn in figure 3.

2.5 Expanding D3-branes, five-form flux and geometric transition

We have seen that the radius R of a spherical D3-brane in AdS_5 with modified boundary condition (labeled by f) on a quadrupole deformation mode of S^5 feels a quartic potential.

[5] If in (2.3) we had left a generic scale μ, (2.38) would have read

$$\int d^4\tilde{x}\, V_{\text{eff}}(\tilde{x}) = -\frac{f}{1 - f\ln(\mu R_{\text{AdS}})} \frac{5\pi^2}{3N^2} \int d^4\tilde{x} \left[\phi_1^2 - \frac{1}{5}\sum_{i=2}^{6}\phi_i^2\right]^2 . \tag{2.41}$$

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For $f > 0$, this potential tries to blow up the D3-brane to infinite radius in finite time. For sufficiently small spherical D3-branes, this is prevented by the positive R^2 term in the potential, corresponding to the conformal coupling of the SYM scalars to the curvature of S^3. For sufficiently large branes, the quartic potential wins and the branes are pushed to the boundary of AdS_5 in finite time.

In which contexts do such large spherical branes play a role? On the one hand, one could start with a system without them and they could be spontaneously created by quantum tunneling. This could happen, for instance, to the pure $AdS_5 \times S^5$ with modified ($f > 0$) boundary conditions, which is known to be only meta-stable (we expect the nucleating spherical branes to be closely related to the instanton solutions of [7]). This is obviously a dynamical process that cannot be described in classical supergravity. On the other hand, one could consider an initial state with large spherical D3-branes present and study the time evolution of this state. This is analogous to the point of view taken in [7, 8], where the interest was in the evolution into a big crunch, not so much in instabilities of pure AdS_5.

To make the analogy between spherical D3-brane evolution and the (super)gravity solutions of [7, 8] more precise, we have to relate the radius R of spherical D3-branes to the scalar field φ appearing in (2.1). As we shall discuss in section 2.5.1, this was done in [4], at least for the related system of flat D3-branes and AdS_5 in Poincaré coordinates. The upshot is that configurations that fit in the consistent truncation (2.1) correspond not to a single D3-brane but to specific distributions of D3-branes, and thus to specific distributions of radii. So to make contact with the supergravity solutions of [7, 8], we should start with such a distribution of large spherical D3-branes.

One point that may appear puzzling at first is related to the five-form flux through the S^5 in the ten-dimensional supergravity solutions of [8]. A spherical D3-brane acts as a domain wall, with the five-form flux inside being one unit smaller than the flux outside. If the big crunch instability in the solutions of [8] corresponds to spherical D3-branes expanding to infinite size, one might thus expect that at any value of the radial coordinate r, the five-form flux should decrease as a function of time as spherical D3-branes expand from radius smaller than r to radius bigger than r. However, if we compute the flux through S^5 for the solutions of [8], we find that the flux remains constant as the bulk scalar field evolves in time:

$$\int_{S^5} \hat{G}_5 = -\int d\xi d\Omega_5 U \Delta^{-2} R_{\rm AdS}^4 = 16\pi^4 g_s \alpha'^2 N \,. \tag{2.42}$$

A related point is that the solutions of [8] solve the supergravity equations of motion without any D3-brane sources present. The resolution of this paradox lies in the concept of geometric transition [2, 3], which relates a situation with D-brane sources explicitly present to a situation with the D-branes replaced by flux (and the location of the D-branes "cut out" of the space).[6] In our spherical D3-brane picture, we considered the shape of the S^5 not to change with time and treated the D3-branes as sources; in the solutions of [8], the D3-branes are not explicitly present, but the shape of the S^5 changes in such a way that the would-be locations of the D3-branes are always "cut out" of the spacetime.

[6] Of course, in the present situation, neither description is valid in the regime in which the space-time is highly curved.

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2.5.1 Relating D3-brane positions with the bulk scalar field

As mentioned earlier in this section, the bulk scalar field φ can be related to D3-brane positions. The near-horizon limit of a distribution σ of parallel D3-branes in n transverse dimensions is given by

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{H}}\left(-dt^2+dx_1^2+dx_2^2+dx_3^2\right)-\sqrt{H}\sum_{i=1}^{6}dy_i^2\,, \\ H &= \int d^n\omega\sigma(\vec{\omega})\frac{R_{\rm AdS}^4}{|\vec{y}-\vec{\omega}|^4}\,. \end{aligned} \tag{2.43}$$

For generic distributions σ, this does not fall in the class of metrics (2.10), so generic D3-brane configurations cannot be described using the consistent truncation (2.1). In [4], it was shown that specific configurations of D3-branes do have near-horizon geometries that can be described using the consistent truncation. For instance, an SO(5) symmetric configuration of D3-branes distributed on a one-dimensional interval of length ℓ according to $\sigma(\vec{\omega})=\frac{2}{\pi\ell^2}\sqrt{\ell^2-|\vec{\omega}|^2}$ gives rise to the metric

$$ds^2=\frac{\xi r^2}{\lambda^3R_{\rm AdS}^2}\left[dx_\mu^2+\frac{R_{\rm AdS}^4}{r^4}\frac{dr^2}{\lambda^6}\right]+\frac{\lambda^3R_{\rm AdS}^2}{\xi}\left[\xi^2d\theta^2+\cos^2\theta d\Omega_4^2\right]\,, \tag{2.44}$$

with

$$\lambda^{12}=1+\frac{\ell^2}{r^2}\,,\qquad \xi^2=1+\frac{\ell^2}{r^2}\cos^2\theta\,. \tag{2.45}$$

Comparing with (2.10), one finds that in the regime of large radial coordinate, the scalar field profile is

$$\varphi(x,r)=\frac{\ell^2}{6\gamma}\frac{1}{r^2}\,, \tag{2.46}$$

with γ defined after (2.1). Therefore, the coefficient of the asymptotic fall-off of φ is directly related to the size over which D3-branes are spread out. This conclusion holds for flat D3-branes and in the Poincaré coordinate system, which appears when taking near-horizon limits. However, since near the boundary of AdS spherical branes are almost flat and Poincaré coordinates are a good approximation to global coordinates, we expect the conclusion to extend to large spherical D3-branes in global AdS.

3 M2-branes in $AdS_4\times S^7/\mathbb{Z}_k$

In this section, we study spherical M2-branes in global $AdS_4\times S^7$ and $AdS_4\times S^7/\mathbb{Z}_k$. M-theory allows a consistent truncation to four-dimensional gravity with a negative cosmological constant coupled to a single scalar field [24]. This scalar corresponds to a quadrupole deformation of the seven-sphere, has a mass that is above the Breitenlohner-Freedman bound [18] and preserves a subgroup of the full SO(8) symmetry. Along the lines of the discussion in section 2, we compute the M2-brane effective potential as a function of the boundary conditions on this bulk scalar field. Therefore, we determine the coupling of the five-dimensional scalar to M2-branes considering the M2-brane action and the consistent truncation ansatz that relates the eleven and four-dimensional solutions. We observe that the modified boundary conditions correspond to adding a cubic boundary interaction to the bulk action and we compute the interaction of the M2-brane with the boundary via this cubic vertex. (Note that, due to the non-linearity of the boundary conditions we will consider, their effect cannot be absorbed in a modification of the scalar field propagator.)

Specifically, we will consider a class of AdS invariant boundary conditions that corresponds to adding a marginal triple trace deformation to the dual field theory [7, 13, 14]. We will first discuss M-theory on $AdS_4 \times S^7$, which is obtained as the near-horizon geometry of M2-branes in flat space and is dual to the $k = 1$ case of ABJM theory. Then we will consider ABJM theory for general k, which corresponds to M2-branes on a $\mathbb{Z}_k$ orbifold of $\mathbb{C}^4$, which have $AdS_4 \times S^7/\mathbb{Z}_k$ as near-horizon geometry.

In section 3.1 we review the bulk setup as well as some relevant aspects of ABJM theory. We identify the deformation that corresponds, according to the AdS/CFT dictionary, to our choice of boundary conditions. In section 3.2, we use the lift to the eleven-dimensional solution to identify the coupling of the bulk scalar field to spherical M2-branes. In section 3.3 we compute the propagator for the four-dimensional scalar field (for standard boundary conditions). In section 3.4, we compute the potential for spherical M2-branes in $AdS_4 \times S^7$. Finally, in section 3.5, we extend the discussion to M2-branes in $AdS_4 \times S^7/\mathbb{Z}_k$ and comment on the 't Hooft limit of the result.

3.1 Setup

M-theory in asymptotically $AdS_4 \times S^7$ spacetimes has four-dimensional SO(8) gauged $\mathcal{N} = 8$ supergravity as its low energy limit. This theory allows a consistent truncation to four-dimensional gravity coupled to a single scalar field that preserves an $SO(4) \times SO(4)$ symmetry

$$S = \frac{V_{S^7}}{\kappa_{11}^2} \int d^4x \sqrt{g} \left[\frac{R}{2} - \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi + \frac{1}{R_{\text{AdS}}^2} \left(2 + \cosh(\sqrt{2}\varphi) \right) \right], \tag{3.1}$$

with $2\kappa_{11}^2 = (2\pi)^8 \ell_p^9$ in terms of the eleven-dimensional Planck length and $V_{S^7} = \pi^4 (2R_{\text{AdS}})^7/3$. The potential has a maximum for vanishing scalar field that corresponds to the AdS_4 vacuum solution. Small fluctuations around the the AdS solution have a mass $m^2 = -2/R_{\text{AdS}}^2$, which is above the BF bound (see footnote 2), and therefore the maximally supersymmetric solution, with the standard boundary conditions, is both perturbatively and non-perturbatively stable. In global coordinates the AdS_4 metric reads

$$ds^2 = -\left(1 + \frac{r^2}{R_{\text{AdS}}^2}\right) dt^2 + \frac{dr^2}{1 + \frac{r^2}{R_{\text{AdS}}^2}} + r^2 d\Omega_2^2 \,. \tag{3.2}$$

In any asymptotically AdS solution, the scalar field behavior at large radial coordinate is

$$\varphi(x, r) = \frac{\alpha(x)}{r} + \frac{\beta(x)}{r^2} \,, \tag{3.3}$$

where x collectively denotes the time coordinate and the S^2 angles. The usual boundary conditions correspond to taking either $\alpha = 0$ (which can be chosen for any m^2) or $\beta = 0$

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(which can be chosen for scalars in the mass range $m^2_{\rm BF} < m^2 < m^2_{\rm BF} + 1/R^2_{\rm AdS}$). There exists however a whole one-parameter family of AdS invariant boundary conditions, i.e., boundary conditions that preserve the asymptotic symmetries of AdS spacetime, which allow the construction of well-defined and finite Hamiltonian generators [19, 25]. The general class is

$$\beta(x) = -h\alpha(x)^2\,, \tag{3.4}$$

where h is an arbitrary constant [25]. For $h \neq 0$, smooth asymptotically AdS initial data can evolve into a big crunch singularity [7].[7]

Adding to the bulk action (3.1) the boundary term

$$S_{\rm bdy} = \frac{V_{S^7}}{\kappa^2_{11} R_{\rm AdS}} \int_\partial d^3x \sqrt{g_{\rm bdy}} \left(-\frac{1}{2}\varphi^2 + \frac{h}{3}\varphi^3 + \frac{h^2}{2}\varphi^4 \right)\,, \tag{3.5}$$

the boundary condition (3.4) follows from a variational principle. As in section 2.1, we have introduced a regularized boundary ∂ in spacetime, located at $r = \Lambda$.

M-theory in asymptotically $AdS_4 \times S^7$ spacetimes with $\beta = 0$ boundary conditions is dual to the three-dimensional superconformal field theory that describes the low energy dynamics of coincident M2-branes. In [13], Aharony, Bergman, Jafferis and Maldacena (ABJM) proposed a specific three-dimensional $\mathcal{N} = 6$ superconformal $\mathrm{U}(N)\times\mathrm{U}(N)$ Chern-Simons-matter theory with levels k and $-k$ as the world-volume theory of N coincident M2-branes on a $\mathbb{C}^4/\mathbb{Z}_k$ singularity. Besides the two $\mathrm{U}(N)$ gauge fields A and $\hat{A}$, the theory contains scalar fields Y^A, $A = 1, \ldots, 4$, transforming in the fundamental representation of the $\mathrm{SU}(4)_R$ R-symmetry group and in the bifundamental $(N, \bar{N})$ of the gauge group. The Hermitean conjugate scalar fields $Y^{A\dagger}$ transform in the anti-fundamental representation of $\mathrm{SU}(4)_R$ and in the $(\bar{N}, N)$ of the gauge group. The action reads

$$\begin{aligned} S_0 = \int d^3x \Big[& \frac{k}{4\pi}\epsilon^{abc}\mathrm{Tr}\left(A_a\partial_b A_c + \frac{2i}{3}A_aA_bA_c - \hat{A}_a\partial_b\hat{A}_c - \frac{2i}{3}\hat{A}_a\hat{A}_b\hat{A}_c \right) \\ & -\mathrm{Tr}(D_aY^A)^\dagger D^aY^A + V_{\rm bos} + \text{terms with fermions} \Big]\,, \end{aligned} \tag{3.6}$$

where $V_{\rm bos}$ is a sextic potential for the scalars and we will not need the fermion fields explicitly in the following. The bulk setup we considered in this section corresponds to the case $k = 1$, for which the transverse space to the M2-branes is simply $\mathbb{R}^8$. We will discuss Chern-Simons level $k > 1$ in section 3.5. In ABJM theory on $\mathbb{R} \times S^2$, the four complex scalars Y^A effectively get mass $m^2 = 1/4$ due to the conformal coupling to the curvature of the S^2 (which we choose to have unit radius; see footnote 3). The boundary condition $\beta = -h\alpha^2$ corresponds to adding a marginal triple trace deformation to the boundary action

$$S = S_0 + \frac{h}{3}\int d^3x \mathcal{O}^3\,. \tag{3.7}$$

Here, $\mathcal{O}$ is the dimension one chiral primary operator

$$\mathcal{O} = c\,Tr\left(Y^1Y_1^\dagger + Y^2Y_2^\dagger - Y^3Y_3^\dagger - Y^4Y_4^\dagger \right)\,, \tag{3.8}$$

[7]The sign of h is irrelevant, as it can be changed by redefining $\varphi \to -\varphi$.

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which preserves the same $\mathrm{SO}(4)\times\mathrm{SO}(4)$ subgroup of $\mathrm{SO}(8)$ as the bulk scalar φ in the consistent truncation. The constant c in (3.8) depends on the two dimensionless parameters N and k in a way that we will determine in section 3.4.

Taking the decoupling limit of a system of coincident M2-branes in eleven-dimensional flat space, we observe that the world-volume field theory of ABJM with $k=1$ has a dual gravitational description in terms of M-theory on $AdS_4\times S^7$. In this description, the eigenvalues of the four complex scalar fields Y^A and of their Hermitean conjugates correspond to M-brane positions in the transversal space as in the case of $\mathcal{N}=4$ SYM (see section 2.3 of [13] for a discussion, and [26] for more details). The deformation (3.7) provides a sextic potential for these positions that is unbounded below and above, whatever the sign of h. This potential is sufficiently strong to make the eigenvalues become infinite in finite time, corresponding to M2-branes reaching the conformal boundary of AdS in finite time. In sections 3.4 and 3.5, we will obtain the effective potential of spherical M2-branes as a function of the boundary conditions in the bulk and show that it matches the deformation (3.7).

3.2 Coupling of the bulk scalar field to spherical M2-branes

The lift of the four-dimensional solution of (3.1) to eleven-dimensional supergravity is given in [27]. Letting $F=e^{\varphi/\sqrt{2}}$ and $\tilde{\Delta}=F\cos^2\theta+F^{-1}\sin^2\theta$, the full eleven-dimensional metric and four-form read

$$ds_{11}^2 = \tilde{\Delta}^{2/3}ds_4^2+4R_{\mathrm{AdS}}^2\left[\tilde{\Delta}^{2/3}d\theta^2+\tilde{\Delta}^{-1/3}\left(F\sin^2\theta d\Omega_3^2+F^{-1}\cos^2\theta d\tilde{\Omega}_3^2\right)\right], \quad (3.9)$$

$$\hat{F}_4 = -\frac{U}{R_{\mathrm{AdS}}}\epsilon_4+8R_{\mathrm{AdS}}\sin\theta\cos\theta F^{-1}*dF\wedge d\theta\,, \quad (3.10)$$

with

$$U=-2-F^2\cos^2\theta-F^{-2}\sin^2\theta\,. \quad (3.11)$$

We have chosen coordinates in terms of which the unit seven-sphere metric would read

$$d\Omega_7^2=d\theta^2+\sin^2\theta d\Omega_3^2+\cos^2\theta d\tilde{\Omega}_3^2\,, \quad (3.12)$$

with $0\le\theta\le\pi$ and, in (3.10), ϵ_4 and $*$ are the four-dimensional volume-form and dual.

We want to repeat the procedure we carried out in section 2.2 and consider a probe M2-brane in the eleven-dimensional lifted solution to determine its coupling to the bulk field φ. The action of the probe brane is

$$S_{M2}=S_{\mathrm{DBI}}+S_{\mathrm{WZ}}=-\tau_2\int d^3x\sqrt{\hat{G}}+\mu_2\int\hat{C}_3\,, \quad (3.13)$$

where $\hat{G}$ is the determinant of the pull-back of the eleven-dimensional metric to the M2-brane worlvolume, $d\hat{C}_3=\hat{F}_4$ and where, for convenience, we have split the M2-brane action in analogy with the conventional notation for D-brane actions . The tension and charge are $\tau_2=\mu_2=2\pi(2\pi\ell_p)^{-3}$. In the static gauge and to linear order in φ, the "DBI" part of the action reads

$$S_{\mathrm{DBI}}=-\tau_2\int d^3x\sqrt{-\hat{g}}\left[1+\frac{1}{\sqrt{2}}\varphi\left(\cos^2\theta-\sin^2\theta\right)+\frac{1}{2}g_{ij}\partial_a x^i\partial^a x^j\right], \quad (3.14)$$

where $\hat{g}$ is the determinant of the pull-back of the four-dimensional metric $g_{\mu\nu}$ to the three-dimensional world-volume, the index a labels the coordinates along the M2-brane world-volume and the index i runs over the eight transverse directions. The Wess-Zumino action is

$$S_{\rm WZ} = \frac{\mu_2}{R_{\rm AdS}} \int_{V_4} d^4x \sqrt{-g} \left[3 + \sqrt{2}\varphi \left(\cos^2\theta - \sin^2\theta \right) \right] , \tag{3.15}$$

expressed as an integral over the four-dimensional volume enclosed by the M2-brane. Here g denotes the determinant of the bulk metric.

Choosing the bulk geometry to be AdS_4 in the global coordinates (3.2) and specializing to a spherical M2-brane of radius R that is localized on S^7, the sources for the scalar field φ are

$$\mathcal{J}_{\rm DBI}(r) = -\frac{\tau_2}{R_{\rm AdS}} \frac{1}{\sqrt{2}} (\cos^2\theta - \sin^2\theta) r \delta(r - R) , \tag{3.16}$$

$$\mathcal{J}_{\rm WZ}(r) = \begin{cases} 2\dfrac{\mu_2}{R_{\rm AdS}} \dfrac{1}{\sqrt{2}} (\cos^2\theta - \sin^2\theta) & r \leq R \\ 0 & r > R . \end{cases} \tag{3.17}$$

To interpret the result we will obtain for the effective potential in the language of ABJM theory and compare it with the deformation (3.7), we introduce canonically normalized scalars on the conformal boundary of metric $d\tilde{s}^2 = -d\tilde{t}^2 + d\Omega_3^2$. The relation between these scalars and the M2-brane radius R and S^7 angles is

$$\begin{aligned} \phi_1 &\equiv 2R_{\rm AdS}\sqrt{\tau_2 R} \cos\theta \cos\Omega_1, \quad \phi_2 \equiv 2R_{\rm AdS}\sqrt{\tau_2 R} \cos\theta \sin\Omega_1 \cos\Omega_2, \quad \ldots \\ \phi_5 &\equiv 2R_{\rm AdS}\sqrt{\tau_2 R} \sin\theta \cos\Omega_4, \quad \phi_6 \equiv 2R_{\rm AdS}\sqrt{\tau_2 R} \sin\theta \sin\Omega_4 \cos\Omega_5, \quad \ldots \end{aligned} \tag{3.18}$$

as can be seen form (3.14). Complex combinations of these fields will correspond to eigenvalues of the fields Y^A appearing in (3.6).

3.3 Propagator of the bulk scalar field

To compute the propagator for the field φ, we follow again [23] and separate variables as

$$\varphi(x, r) = e^{-i\omega t} Y_{\ell,m}(\Omega) \Psi(r) , \tag{3.19}$$

where the spherical harmonics satisfy $\nabla^2_{S^2} Y_\ell = -\ell(\ell+1) Y_\ell$, with $\ell \geq 0$. The radial solution that is regular in the interior, in the sense of (2.33), is

$$\Psi_1(V) = (1 - V) V^{\ell/2} \, {}_2F_1\left(a, b, a + b - \frac{1}{2}; V \right) . \tag{3.20}$$

The propagator is constructed from (3.20) and the radial solution with asymptotic behavior

$$\Psi_2(V) = (1 - V)^{1/2} V^{\ell/2} \left[{}_2F_1\left(a - \frac{1}{2}, b - \frac{1}{2}, \frac{1}{2}; 1 - V \right) + K_\infty \, {}_2F_1\left(a, b, \frac{3}{2}; 1 - V \right) \right] , \tag{3.21}$$

in terms of a coefficient K_∞ that implements the specific choice of boundary conditions. The standard supersymmetric choice $\beta = 0$ sets $K_\infty = 0$. In (3.20) and (3.21), we have again denoted $a = 1 + \frac{1}{2}(\ell + \omega)$, $b = 1 + \frac{1}{2}(\ell - \omega)$ and $V = r^2/(R_{\rm AdS}^2 + r^2)$.

Combining the two solutions above with the appropriate normalization factor, we obtain the Feynman propagator

$$G_F(x,V;x',V') = -\frac{\kappa_{11}^2}{R_{\rm AdS}V_{S^7}}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\sum_{\ell,m}\frac{\Gamma(a-\frac{1}{2})\Gamma(b-\frac{1}{2})}{2\sqrt{\pi}\Gamma(a+b-\frac{1}{2})}e^{-i\omega(t-t')}\times \\ Y_{\ell,m}(\Omega)Y_{\ell,m}(\Omega')\left[\theta(V'-V)\Psi_1(V)\Psi_2(V')+\theta(V-V')(V\leftrightarrow V')\right]. \tag{3.22}$$

3.4 M2-brane effective potential

In this section, we compute the effective potential for the radial coordinate R of a probe M2-brane that extends along a two-sphere and is localized on S^7. We evaluate the M2-brane action (3.13) in an $AdS_4 \times S^7$ background.[8] The BPS relation between the charge and tension of the brane guarantees the cancellation of the leading order terms in the radial coordinate, as can be seen from (3.14) and (3.15). The h-independent term that survives the cancellation is an attractive potential linear in R, which corresponds to the conformal coupling of the scalar fields Y^A of the dual theory on $\mathbb{R}\times S^2$. The dependence of the potential on the modified boundary conditions shows up, to lowest order, in a Feynman diagram in which the probe brane interacts with the boundary, exchanging scalar φ modes through the cubic coupling in (3.5). Generalizing (2.39) to a cubic boundary interaction, we obtain

$$S_{\rm eff} = \frac{h}{3}\frac{\tau_2^3\kappa_{11}^4 R_{\rm AdS}}{V_{S^7}^2}\int d^3x R^3\frac{1}{2\sqrt{2}}\left(\cos^2\theta-\sin^2\theta\right)^3 . \tag{3.23}$$

In terms of the boundary fields of equation (3.18) and of the background metric $\tilde{g}$ defined above (3.18), the result becomes

$$\int d^4\tilde{x}\, V_{\rm eff}(\tilde{x}) = -\frac{h}{N^3}\frac{3\pi^2}{8}\int d^4\tilde{x}\left[\sum_{i=1}^{4}\phi_i^2-\sum_{i=5}^{8}\phi_i^2\right]^3 . \tag{3.24}$$

In the last step we have used the relation $2R_{\rm AdS}/\ell_p = (2^5\pi^2N)^{1/6}$ [13], which relates the radius of AdS with N units of flux to the eleven-dimensional Planck length. For a non-vanishing value of h, this is a sextic potential with unstable directions. Using a similar argument as after (2.38), one can see that the potential (3.24) matches the deformation (3.7) of ABJM theory. For k=1, it fixes the N-dependence of the operator $\mathcal{O}$in (3.8) to be $c\sim 1/N$.

3.5 Extension to $k>1$

The result of the previous section corresponds to the $k=1$ case of ABJM theory. To generalize to arbitrary k, consider the $\mathbb{Z}_k$ orbifold of the eleven-dimensional supergravity solution. In [13, 28], the metric on the seven-sphere was written in a Hopf-fibered way:

$$ds^2_{S^7} = (d\chi+\omega)^2 + ds^2_{\mathbb{C}P^3} \tag{3.25}$$

with χ periodic with periodicity 2π. The $\mathbb{Z}_k$ action simply changes the periodicity of the coordinate χ to $2\pi/k$. Since the volume of the quotient space is smaller by a factor k than

[8] Actually the requirement of an asymptotically $AdS_4\times S^7$ background is sufficient.

the original one, in order to have N units of flux of the four-form (3.10) on the quotient space, we need to start with $N' = kN$ units on the covering space. The circle labeled by χ can be interpreted as the M-theory circle.

In the parametrization (3.12) of S^7, we can exhibit the χ direction by writing

$$d\Omega_3^2 = [d(\chi + \tilde{\chi}) + \omega]^2 + ds^2_{\mathbb{C}P^1}\,, \quad d\tilde{\Omega}_3^2 = [d(\chi - \tilde{\chi}) + \tilde{\omega}]^2 + d\tilde{s}^2_{\mathbb{C}P^1}\,. \tag{3.26}$$

where as before χ has periodicity $2\pi/k$ after the $\mathbb{Z}_k$ identification.

The $\mathbb{Z}_k$ identification on the χ direction rescales the volume of the S^7, V_{S^7}, by a factor $1/k$. As pointed out in [14], the bulk scalar field φ survives the $\mathbb{Z}_k$ quotient, so, to extend the previous discussion to an arbitrary value of the Chern-Simons level, it suffices to trace back its contributions in the computation of the effective potential. As a consequence of the orbifolding, the actions (3.1) and (3.5) get rescaled by a factor $1/k$ and therefore, the propagator for the field φ (3.22) has to be multiplied by a factor k. The M2-brane action (3.13) is unaffected by the identification since the M-theory direction is transverse to the M2-brane. The overall effect of the $\mathbb{Z}_k$ action is to rescale the final result (3.24) by a factor k^2. Substituting $N' = kN$, it combines into

$$\int d^4\tilde{x}\, V_{\rm eff}(\tilde{x}) = -\frac{h}{kN^3}\frac{3\pi^2}{8}\int d^4\tilde{x}\left[\sum_{i=1}^{4}\phi_i^2 - \sum_{i=5}^{8}\phi_i^2\right]^3\,. \tag{3.27}$$

We now discuss various parameter regimes of the theory of $N' = kN$ M2-branes on a $\mathbb{C}^4/\mathbb{Z}_k$ singularity to comment the k and N dependence of the effective potential. As discussed in [13], the radius of the M-theory circle in Planck units is of order $R_{\rm AdS}/k\ell_p \sim (kN)^{1/6}/k$, while the radius of the $\mathbb{C}P^3$ factor is always large in Planck units if $kN \gg 1$. Thus the M-theory description reduces to a weakly coupled type IIA string theory whenever $k^5 \gg N$. In this limit, the M2-action (3.13) reduces to the action of a D2-brane in an $AdS_4 \times \mathbb{C}P^3$ background. Due to the presence of two dimensionless parameters N and k, we can also define a 't Hooft coupling $\lambda \equiv N/k$ and consider a 't Hooft limit $N \to \infty$ with λ fixed. The radius of curvature in string units is of order $\lambda^{1/4}$, so the supergravity description is valid if $\lambda \gg 1$. In this 't Hooft limit, M-theory (or eleven-dimensional supergravity) always reduces to weakly coupled type IIA string theory (or supergravity), and the spherical M2-branes are really D2-branes.

From (3.27), we can infer that the operator $\mathcal{O}$ in (3.8) scales like $c \sim (kN^3)^{-1/3}$. Since N/k is fixed as $N \to \infty$, the $1/kN^3$ dependence of the triple trace deformation of ABJM theory precisely agrees with the $1/N^4$ scaling assumed in [14], based on the requirement that the 't Hooft limit should exist and be non-trivial.

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A Brane effective potentials in the Poincaré patch

The computations of D-brane effective potentials in the main text were done for spherical D-branes in global AdS space-times. In this appendix, we discuss the analogous computations for flat D-branes in the Poincaré patch of AdS.

In Poincaré coordinates, the AdS_{d+1} metric reads

$$ds^2 = \frac{R_{\rm AdS}^2}{\rho^2}\left(-dt^2 + d\rho^2 + d\vec{x}^2\right) , \tag{A.1}$$

where $d\vec{x}^2$ is the flat metric on $\mathbb{R}^{d-1}$ and $0 \le \rho \le \infty$. In this parameterization, the spacetime has an horizon at $\rho = \infty$ and the conformal boundary at $\rho = 0$ is $\mathbb{R}^{d-1}$.

Expanding a free massive scalar field in Minkowski plane waves,

$$\varphi(\vec{x}, \rho) = e^{-i\omega t + i\vec{k}\cdot\vec{x}} \rho^{d/2} \Psi(\rho) , \tag{A.2}$$

the radial wave equation becomes

$$\rho^2 \partial_\rho^2 \Psi + \rho \partial_\rho \Psi - \left[m^2 + \frac{d^2}{4} + \rho^2 \left(\vec{k}^2 - \omega^2 \right) \right] \Psi = 0 . \tag{A.3}$$

For $q^2 = \vec{k}^2 - \omega^2 > 0$, the two solutions are [23]

$$\Psi_1^+(\rho) = K_\nu(q\rho) , \qquad \Psi_2^+(\rho) = I_\nu(q\rho) , \tag{A.4}$$

with $\nu = \frac{1}{2}\sqrt{d^2 + 4m^2 R_{\rm AdS}^2}$. In the mass range $m_{\rm BF}^2 \le m^2 < m_{\rm BF}^2 + 1/R_{\rm AdS}^2$ we are interested in, corresponding to $0 \le \nu < 1$, both solutions are normalizable at the boundary of spacetime, while only Ψ_1^+ is regular in the interior in the sense of (2.33).[9] In our computation of the D-brane effective potential, one would expect that only the $q^2 = 0$ modes contribute. However, we will see that it is useful to consider a regulator momentum $q_0^2 > 0$. For $q^2 > 0$, we construct the propagator starting from the solution that is regular at the origin and from a solution with specified behavior near the boundary:

$$\Psi_1(\rho) = K_\nu(q\rho) , \qquad \Psi_2(\rho) = I_\nu(q\rho) + C_\infty^P K_\nu(q\rho) , \tag{A.7}$$

[9] For $q^2 < 0$, the solutions are

$$\Psi_{1/2}^-(\rho) = J_{\pm\nu}(|q|\rho) , \tag{A.5}$$

when ν is non integer and

$$\Psi_1^-(\rho) = J_\nu(|q|\rho) , \qquad \Psi_2^-(\rho) = Y_\nu(|q|\rho) \tag{A.6}$$

for integer ν. Regularity in the interior selects a Hankel function, while for $0 \le \nu < 1$ both solutions are normalizable near the boundary.

where C_∞^P will be chosen such that Ψ_2 satisfies the boundary conditions of interest. The Feynman propagator then reads

$$G_F(\vec{x},\rho;\vec{x}',\rho') = -\frac{\kappa_D^2}{R_{\rm AdS}^{(d-1)} V_{S^{D-(d+1)}}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^d\vec{k}}{(2\pi)^d} e^{-i\omega(t-t')+i\vec{k}\cdot(\vec{x}-\vec{x}')} \times \\ \rho^{d/2}\rho'^{d/2}\{\theta(\rho-\rho')\Psi_1(\rho)\Psi_2(\rho')+\theta(\rho'-\rho)(\rho\leftrightarrow\rho')\}\,, \tag{A.8}$$

where $D = 10$ (or $D = 11$) respectively for type IIB supergravity (or eleven-dimensional supergravity). We are now ready to specialize to the two cases of interest in this paper. Consider a probe D3-brane (or M2-brane) extended in flat four-dimensional (or three-dimensional) space sitting at a radial Poincaré coordinate $\bar{\rho}$ and localized at a point in S^5 (or S^7).

In the five-dimensional setup of section 2, $\nu = 0$ and the source terms are

$$\mathcal{J}_{\rm DBI}(\rho) = 5\gamma\frac{\tau_3}{R_{\rm AdS}}\left(\cos^2\xi - \frac{1}{5}\sin^2\xi\right)\rho\,\delta(\rho-\bar{\rho})\,, \tag{A.9}$$

$$\mathcal{J}_{\rm WZ}(\rho) = \begin{cases} -10\gamma\dfrac{\mu_3}{R_{\rm AdS}}\left(\cos^2\xi - \dfrac{1}{5}\sin^2\xi\right) & \rho\geq\bar{\rho} \\ 0 & \rho<\bar{\rho}\,. \end{cases} \tag{A.10}$$

The propagator satisfying the boundary conditions (2.5) defined at the scale μ appearing in (2.3), has

$$C_\infty^P = \frac{f}{1+f\left(\gamma_E + \ln\frac{q}{2\mu}\right)}\,, \tag{A.11}$$

where γ_E is again Euler's constant. Here we see why it is useful to introduce a regulator $q^2 = q_0^2 > 0$: for $q^2 = 0$, we would have found an infrared divergent expression (we will comment more on this below). The (regularized) effective potential computed as in (2.38) is

$$\int d^4x\, V_{\rm eff}(x) = -\frac{f}{1+f\left(\gamma_E+\ln\frac{q_0}{2\mu}\right)}\frac{5\pi^2}{3N^2}\int d^4x\left[\phi_1^2 - \frac{1}{5}\sum_{i=2}^{6}\phi_i^2\right]^2\,, \tag{A.12}$$

where we have introduced the fields

$$\phi_1 \equiv \sqrt{\tau_3}\frac{R_{\rm AdS}^2}{\bar{\rho}}\cos\xi, \quad \phi_2 \equiv \sqrt{\tau_3}\frac{R_{\rm AdS}^2}{\bar{\rho}}\sin\xi\cos\Omega_1, \quad \ldots \tag{A.13}$$

with canonical kinetic term

$$S_{\rm kin} = -\frac{1}{2}\int d^4x\,\partial_\alpha\phi_i\partial^\alpha\phi_i\,. \tag{A.14}$$

We can now explain what is the role of the IR regulator q_0^2. Since the sources do not depend on t and $\vec{x}$, only the $q^2 = 0$ modes should contribute to the effective potential. As is easy to see by letting $q_0 \to 0$ in (A.12), this would formally give a vanishing result. From a dual field theory point of view, this can be understood as follows. The scale μ

corresponds to the scale at which the coupling constant f is defined, while q is the scale at which the (renormalized) four-point function is computed. In the planar (large N) limit, factorizable diagrams such figure 3 can be resummed and give rise to the running coupling $f/[1+f(\gamma_E+\ln(q_0/2\mu))]$ appearing in (A.12). Note in particular that the formal vanishing of the coupling for $q_0 \to 0$ is not reliable: for $f > 0$, the coupling becomes strong as one flows to the IR and formally becomes infinite at some finite value of q_0, before $q_0 = 0$ is reached. Note also that these infrared divergences were absent in section 2.4, since there the radial position of the brane was effectively massive (corresponding to the conformal coupling to the curvature of S^3 in SYM theory on $\mathbb{R} \times S^3$).

In the four-dimensional case, $\nu = 1/2$ and the sources read

$$\mathcal{J}_{\rm DBI}(\rho) = -\frac{\tau_2}{R_{\rm AdS}}\frac{1}{\sqrt{2}}\left(\cos^2\theta - \sin^2\theta\right)\rho\,\delta(\rho-\bar\rho)\,, \tag{A.15}$$

$$\mathcal{J}_{\rm WZ}(\rho) = \begin{cases} 2\dfrac{\mu_2}{R_{\rm AdS}}\dfrac{1}{\sqrt{2}}(\cos^2\theta - \sin^2\theta) & \rho \ge \bar\rho \\ 0 & \rho < \bar\rho\,. \end{cases} \tag{A.16}$$

The supersymmetric boundary condition sets

$$C^P_\infty = \frac{2}{\pi}\,. \tag{A.17}$$

The result for the effective potential computed as in (3.24) is

$$\int d^4x\, V_{\rm eff}(x) = -\frac{h}{N^3}\frac{3\pi^2}{8}\int d^4x\left[\sum_{i=1}^4 \phi_i^2 - \sum_{i=5}^8 \phi_i^2\right]^3\,, \tag{A.18}$$

in terms of the canonically normalized scalars

$$\begin{aligned} \phi_1 &\equiv 2\sqrt{\frac{\tau_2 R^3_{\rm AdS}}{\bar\rho}}\cos\theta\cos\Omega_1, \quad \phi_2 \equiv 2\sqrt{\frac{\tau_2 R^3_{\rm AdS}}{\bar\rho}}\cos\theta\sin\Omega_1\cos\Omega_2, \quad \dots \\ \phi_5 &\equiv 2\sqrt{\frac{\tau_2 R^3_{\rm AdS}}{\bar\rho}}\sin\theta\cos\Omega_4, \quad \phi_6 \equiv 2\sqrt{\frac{\tau_2 R^3_{\rm AdS}}{\bar\rho}}\sin\theta\sin\Omega_4\cos\Omega_5, \quad \dots \end{aligned} \tag{A.19}$$

The final result does not depend on the regulator q_0. This is in agreement with the fact that the boundary conditions (3.4) are AdS invariant and that in the planar limit the corresponding multi-trace deformation is exactly marginal and preserves conformal invariance.

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Multitrace deformation of the Aharony-Bergman-Jafferis-Maldacena theory

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Motivated by the study of big crunch singularities in asymptotically AdS_4 space-times, we consider a marginal triple trace deformation of Aharony-Bergman-Jafferis-Maldacena (ABJM) theory. The deformation corresponds to adding a potential which is unbounded below. In a 't Hooft large N limit, the beta function for the triple trace deformation vanishes, which is consistent with the near-boundary behavior of the bulk fields. At the next order in the $1/N$ expansion, the triple trace couplings exhibit nontrivial running, which we analyze explicitly in the limit of zero 't Hooft coupling, in which the model reduces to an $O(N) \times O(N)$ vector model with large N. In this limit, we establish the existence of a perturbative UV fixed point, and we comment on possible nonperturbative effects. We also show that the bulk analysis leading to big crunch singularities extends to the $\mathbb{Z}_k$ orbifold models dual to ABJM theory.

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I. INTRODUCTION AND SUMMARY

M theory compactified on S^7 with asymptotically AdS_4 boundary conditions allows a consistent truncation to four-dimensional supergravity with a negative cosmological constant and a single scalar field whose negative mass squared lies just above the Breitenlohner-Freedman bound. Besides the usual supersymmetric boundary conditions, there is a different set of possible, well-defined boundary conditions that break supersymmetry but preserve all anti-de Sitter (AdS) symmetries. In [1] it was shown that the theory with nonsupersymmetric, AdS-invariant boundary conditions admits solutions where smooth, asymptotically AdS initial data evolve into a big crunch singularity—a spacelike singularity that reaches the boundary of AdS_4 in finite global time.

The holographic dual to M theory in asymptotically $AdS_4 \times S^7$ space-times is the three-dimensional superconformal field theory that describes the low energy dynamics of coincident $M2$-branes [2]. This theory can be thought of as living on the boundary of AdS_4. Adopting nonsupersymmetric but AdS-invariant boundary conditions for the bulk corresponds to adding a marginal triple trace potential to the boundary theory. With this correction, the tree-level potential of the boundary theory no longer has a minimum, indicating that the Hamiltonian of the quantum boundary theory may be unbounded below. In Refs. [1,3], the suggestion was made that one might be able to learn something about cosmological singularities in the bulk by studying field theories with potentials which are unbounded below.

At the time of [1,3], however, not much was known about the $M2$-brane theory, even without the unstable deformation. It arises as the infrared (strong coupling) limit of the super-Yang-Mills (SYM) theory living on $D2$-branes, but this infrared limit was hard to describe explicitly. For instance, its spectrum of chiral operators was derived not through field theory computations, but by using the AdS/CFT correspondence and the known Kaluza-Klein spectra of 11-dimensional supergravity compactified on S^7 [4]. But without explicit knowledge of the dual theory, one could not perform reliable field theory computations to give information about cosmological singularities.

A more specific criticism of [1] was raised in [5], where it was argued, based on an analogy with the $O(N)$ vector model at large N, that the deformation of the conformal field theory is marginally irrelevant. Since the behavior of the potential for large field values would then depend on an unknown ultraviolet completion of the theory, it was argued that the unbounded below nature of the potential might be an artifact of the tree-level approximation and, in particular, could be absent in the full quantum theory.

For these reasons, we have recently studied related $AdS_5 \times S^5$ models [6], also suggested in [1]. In these models, the undeformed dual field theory is $\mathcal{N} = 4$ SYM in four dimensions, which is very well understood. The deformation corresponds to adding a negative, unbounded double trace potential [7–9]. In this theory, the coupling of the negative double trace deformation is asymptotically free in the large N limit [8], which we used to argue that the quantum effective potential is unbounded below (the argument for a single scalar field was given in [10]). The relevant coupling becomes arbitrarily small in the regime of interest for studying the cosmologi-

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cal singularity (namely large fields in the boundary theory), rendering perturbation theory more and more reliable as the singularity approaches.

Recently, a concrete proposal for the theory of N coincident $M2$-branes was put forward in Ref. [11] [Aharony-Bergman-Jafferis-Maldacena (ABJM)]. The ABJM theory is an $\mathcal{N} = 6$ superconformal $U(N) \times U(N)$ Chern-Simons theory with levels k and $-k$, respectively. For $k > 1$, the $M2$-branes are localized at the fixed point of a $\mathbb{Z}_k$ orbifold of Minkowski space. The presence of the two parameters N and k allows one to define a 't Hooft limit $N \to \infty$ with N/k fixed. In this regime, one of the S^7 dimensions in the 11-dimensional bulk becomes small and the theory is best described by type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{C}P_3$.

In the present paper, we revisit the issues mentioned above in the context of the ABJM theory. Unlike the model studied in [6], in this case the triple trace potential does not have a definite sign, so the $O(N)$ vector model analogy does not apply directly. Therefore, we introduce a tricritical $O(N) \times O(N)$ vector model as a better analogue and study its fixed point structure—in fact, in the limit of weak 't Hooft coupling ($N/k \to 0$), our deformation of ABJM theory precisely reduces to the $O(2N^2) \times O(2N^2)$ vector model at large N. In this limit, we find that the perturbative beta functions for the various sextic couplings vanish. (A similar scale independence at strong 't Hooft coupling can be inferred from the near-boundary behavior of the corresponding bulk scalar [1].) At the next order in the $1/N$ expansion, the beta functions are nontrivial and the $O(N) \times O(N)$ vector model has several nontrivial UV fixed points, one of which corresponds to the UV regime of our potential with indefinite sign. This shows that at least within perturbation theory and at sufficiently weak 't Hooft coupling, as in the negative double trace deformation of the $\mathcal{N} = 4$ SYM case, the coupling of a negative triple trace deformation is asymptotically free and the quantum effective potential is unbounded below. Parenthetically, let us mention that large N nonperturbative effects are known to destabilize the model in the UV when a time-independent, static system is considered [see [12] for the $O(N)$ vector model and [13] for the $O(N) \times O(N)$ vector model]. However, preliminary analysis indicates that these instabilities are in fact consistent with the time-dependent, cosmological applications we have in mind [14].

In this paper, we also extend the bulk analysis of [1] (which would correspond to $k = 1$) to $k > 1$, which is necessary to make contact with the 't Hooft regime in which we do the field theory analysis. In particular, we show that the scalar field present in the consistent truncation in [1] survives the $\mathbb{Z}_k$ orbifolding, so that the four-dimensional analysis of [1] also holds for $k > 1$.

With these results in hand, one can attempt to define unitary evolution in these theories by using self-adjoint extensions. It will then be interesting to study the implications of this for the nature of cosmological singularities in the bulk. The results of this work will appear elsewhere [14].

The structure of this paper is as follows. In Sec. II, we briefly review the work of [1] on big crunch solutions of AdS supergravity coupled to a scalar field with nonsupersymmetric boundary conditions. In Sec. III, we review the ABJM theory of coincident $M2$-branes and its gravity dual. In Sec. IV, we propose a triple trace deformation dual to the modified boundary conditions in the bulk. By studying the properties of the $O(N) \times O(N)$ vector model under renormalization group flow, we show that, at least in the weak 't Hooft coupling limit, the couplings of the triple trace potential have a perturbative UV fixed point. We discuss $\mathbb{Z}_k$ orbifolds of the bulk models of [1] and show that the scalar field of interest survives the orbifolding.

II. ADS COSMOLOGY

M theory in asymptotically $\mathrm{AdS}_4 \times S^7$ space-times has $D = 4$, $\mathcal{N} = 8$ gauged supergravity as its low energy limit. This theory contains the graviton, 28 gauge bosons in the adjoint of $SO(8)$, and 70 real scalars as its bosonic degrees of freedom. It allows a consistent truncation to four-dimensional gravity coupled to a single scalar field [15]:

$$S = \int d^4x\sqrt{g}\left[\frac{1}{2}R - \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{R_{\mathrm{AdS}}^2}(1 + 2\cosh(\varphi))\right], \tag{2.1}$$

where we have chosen units in which the 4D Planck mass is unity.[1] The maximum of the potential at $\varphi = 0$ corresponds to the AdS_4 vacuum solution. Small fluctuations around the maximum of the potential have $m^2 = -2R_{\mathrm{AdS}}^{-2}$, which is above the Breitenlohner-Freedman bound $m_{BF}^2 = -\frac{9}{4}R_{\mathrm{AdS}}^{-2}$ [16]. Hence, the AdS_4 solution is perturbatively stable. The positive mass theorem implies that, with the usual supersymmetric boundary conditions, it is also nonperturbatively stable. As we shall now review, this need not be the case with other AdS-invariant boundary conditions [1].

In global coordinates, the AdS_4 metric reads

$$ds^2 = R_{\mathrm{AdS}}^2\left(-(1 + r)^2dt^2 + \frac{dr^2}{1 + r^2} + r^2d\Omega_2\right). \tag{2.2}$$

In asymptotically AdS solutions, the scalar field φ decays at large radial coordinate as

[1]This truncation corresponds to setting $\phi^{(12)} = 0$, $\phi^{(13)} = \phi^{(14)} = \varphi$, and the gauge fields to zero in Eq. (2.11) of [15]. As can be seen from (2.12) of [15], this choice preserves $SO(4) \times SO(2) \times SO(2)$. Truncations preserving $SO(6) \times SO(2)$ and $SO(4) \times SO(4)$ are also possible.

$$\varphi \sim \frac{\alpha(t,\Omega)}{r} + \frac{\beta(t,\Omega)}{r^2}. \quad (2.3)$$

The usual boundary conditions correspond to taking either $\alpha = 0$ (which can be chosen for any m^2) or $\beta = 0$ (which can be chosen for scalars in the mass range $-\frac{9}{4}R_{\mathrm{AdS}}^{-2} < m^2 < -\frac{5}{4}R_{\mathrm{AdS}}^{-2}$). These are in fact two special cases out of a one-parameter class of boundary conditions that are anti-de Sitter invariant and allow the construction of well-defined and finite Hamiltonian generators for all elements of the anti-de Sitter algebra [17,18]. The more general boundary conditions are given by

$$\beta = -h\alpha^2, \quad (2.4)$$

where h is an arbitrary constant [17]. For $h > 0$, solutions were found [1] in which smooth asymptotically AdS initial data evolved into a big crunch, a spacelike singularity reaching the boundary of AdS in finite global time.[2]

M theory in asymptotically $\mathrm{AdS}_4 \times S^7$ space-times with $\beta = 0$ boundary conditions is dual to the three-dimensional superconformal field theory describing the low energy dynamics of coincident $M2$-branes. The scalar mode $\alpha(t,\Omega)$ of a bulk solution corresponds to the expectation value of the dual operator $\mathcal{O}$ in the boundary theory. (Choosing a fixed $\beta \neq 0$ would correspond to adding a source term $\int \beta\mathcal{O}$ to the action of the boundary theory.) The bulk scalar field φ is dual to an operator $\mathcal{O}$ of dimension 1, transforming in the traceless symmetric two-tensor representation of $SO(8)$. In general, adding a term $-\int W(\mathcal{O})$ to the action of the dual field theory corresponds in the bulk theory to adopting modified boundary conditions $\beta(\alpha)$ such that $\beta = W'(\alpha)$ [8,9]. Taking boundary conditions (2.4) is therefore dual to adding a marginal triple trace operator to the boundary action

$$S \to S + \frac{h}{3}\int \mathcal{O}^3. \quad (2.5)$$

In Sec. IV, we shall see that the operator $\mathcal{O}$ can take arbitrarily large positive and negative values, so that the potential we have added is unbounded below for any non-zero value of h. This is consistent with our earlier comment in Footnote 2.

Unlike the asymptotically $\mathrm{AdS}_5 \times S^5$ model studied in [6], the asymptotic behavior (2.3) with (2.4) does not involve a logarithmic dependence on the radial coordinate.[3] In the dual field theory, this corresponds to the fact that conformal invariance is preserved to leading order in $1/N$, which we shall discuss in Sec. IV.

III. ABJM THEORY

Recently, Aharony, Bergman, Jafferis, and Maldacena (ABJM) introduced an $\mathcal{N} = 6$ superconformal $U(N) \times U(N)$ Chern-Simons-matter theory with levels k and $-k$, respectively. The two $U(N)$ gauge fields are denoted A_μ and $\hat{A}_\mu$. The theory contains scalar fields Y^A, $A = 1, \ldots 4$ transforming in the fundamental representation of an $SU(4)$ R symmetry. (Here, we are using the notation of [19].) Each Y^A transforms in the bifundamental $(N, \bar{N})$ representation of the gauge group. The Hermitean conjugate scalar fields $Y_A^\dagger$ transform in the antifundamental representation of $SU(4)$ and the $(\bar{N}, N)$ of the gauge group. We will not need the fermionic fields explicitly in this paper.

The action reads

$$S = \int d^3x \bigg[\frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \mathrm{Tr}\Big(A_\mu \partial_\nu A_\lambda + \frac{2i}{3} A_\mu A_\nu A_\lambda - \hat{A}_\mu \partial_\nu \hat{A}_\lambda - \frac{2i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda \Big) - \mathrm{Tr}(D_\mu Y^A)^\dagger D^\mu Y^A + V_{\mathrm{bos}} + \text{terms with fermions} \bigg], \quad (3.1)$$

with

$$V_{\mathrm{bos}} = -\frac{4\pi^2}{3k^2} \mathrm{Tr}[Y^A Y_A^\dagger Y^B Y_B^\dagger Y^C Y_C^\dagger + Y_A^\dagger Y^A Y_B^\dagger Y^B Y_C^\dagger Y^C + 4 Y^A Y_B^\dagger Y^C Y_A^\dagger Y^B Y_C^\dagger - 6 Y^A Y_B^\dagger Y^B Y_A^\dagger Y^C Y_C^\dagger]. \quad (3.2)$$

The proposal of [11] is that this theory is the world-volume action for N coincident $M2$-branes on a $\mathbb{Z}_k$ orbifold of $\mathbb{C}^4$, with the generator of $\mathbb{Z}_k$ acting as

$$y^A \to \exp(2\pi i/k) y^A \quad (3.3)$$

on complex coordinates y^A. The coupling constant of the ABJM theory is $1/k$. We will be interested in the "'t Hooft" limit of large N with N/k fixed. In this limit, the theory is weakly coupled for $k \gg N$ and strongly coupled for $k \ll N$.

The gravity dual of this system of $M2$-branes is a $\mathbb{Z}_k$ orbifold of $\mathrm{AdS}_4 \times S^7$. Before orbifolding, the $\mathrm{AdS}_4 \times S^7$ solution of 11-dimensional supergravity with N' units of four-form flux reads

$$ds^2 = \frac{R^2}{4} ds^2_{\mathrm{AdS}_4} + R^2 ds^2_{S^7}; \quad (3.4)$$

$$F_4 \sim N' \epsilon_4; \quad (3.5)$$

$$\frac{R}{l_p} = (32\pi^2 N')^{1/6}, \quad (3.6)$$

where $ds^2_{\mathrm{AdS}_4}$ and $ds^2_{S^7}$ have unit radius.

[2]The restriction to $h > 0$ is not essential. In the solutions of [1], the α coefficient of the initial profile of the bulk scalar field is positive. Since the potential of the bulk scalar field in (2.1) is even in φ, there exist similar solutions where α is negative in the initial profile, and β positive. These are solutions with $h < 0$, for which the scalar field becomes negative towards the interior of the bulk.

[3]We note that the absence of a φ^3 interaction in the potential in (2.1) is a necessary condition for there to be no logarithmic tails in the asymptotic profile of $m^2 = -2$ scalars [18].

The $\mathbb{Z}_k$ identification, which acts on the S^7 as in (3.3), preserves an $SU(4) \times U(1)$ subgroup of the isometry group of S^7. It is convenient to rewrite the unit seven-sphere as an S^1 fibration over $\mathbb{C}P^3$:

$$ds^2_{S^7} = (d\chi + \omega)^2 + ds^2_{CP^3}, \tag{3.7}$$

where χ has period 2π and ω is a connection on a topologically nontrivial $U(1)$ bundle on $\mathbb{C}P^3$ [20]. The $\mathbb{Z}_k$ identification simply changes the period of χ to $2\pi/k$. In order to have N units of flux on the quotient space, we choose $N' = kN$. While the radius of the $\mathbb{C}P^3$ factor in (3.4) is always large in Planck units if $kN \gg 1$, the radius of the χ circle in Planck units is of order $R/kl_p \sim (kN)^{1/6}/k$, which is very small in the 't Hooft limit. Therefore, the appropriate description in this regime is as a weakly coupled type IIA string theory. The radius of curvature in string units turns out to be of order $(N/k)^{1/4}$, so the bulk is stringy when the 't Hooft coupling is small.

IV. A TRIPLE TRACE DEFORMATION OF ABJM THEORY

The consistent truncation (2.1) of $D = 4$, $\mathcal{N} = 8$ gauged supergravity was introduced in [15]. The bulk scalar φ, which corresponds to a specific quadrupole deformation of S^7 and transforms as a symmetric traceless tensor under $SO(8)$, is invariant under independent $U(1)$ rotations of the four complex coordinates y^A [see Eq. (2.9) of [15]], and, in particular, under the identification (3.3). This implies that φ survives the $\mathbb{Z}_k$ quotient, so that the bulk analysis of [1] extends to $k > 1$, in particular, to the 't Hooft limit of interest in the present paper. The operator $\mathcal{O}$ is a dimension one chiral primary operator with the same symmetry properties as φ under the preserved $SU(4)$ subgroup of $SO(8)$. A natural candidate is

$$\mathcal{O} = \frac{1}{N^2}\,\mathrm{Tr}(Y^1 Y_1^\dagger - Y^2 Y_2^\dagger). \tag{4.1}$$

To understand the factor $1/N^2$ in (4.1), note that in general the large N limit of theories of matrix-valued fields Φ is taken as follows (see for instance [8]). Trace operators are normalized as $\mathcal{O} = \mathrm{Tr}F(\Phi)/N$ and the action has the form $N^2 W(\mathcal{O})$, where neither F nor W depend explicitly on N. The fields Y appearing in the action (3.1) are rescaled to have an N-independent kinetic term in the 't Hooft limit: $Y \sim \sqrt{N}\Phi$, which explains the extra factor of $1/N$ in (4.1). The triple trace vertex appearing in the deformation (2.5) is drawn in 't Hooft double line notation in Fig. 1. In terms of a coupling f that is kept fixed as the 't Hooft limit is taken, we have added to the single trace potential (3.2) a triple trace term

$$V = -\frac{f}{N^4}[\mathrm{Tr}(Y^1 Y_1^\dagger - Y^2 Y_2^\dagger)]^3, \tag{4.2}$$

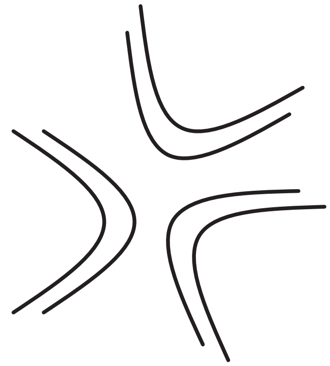

FIG. 1. The $[\mathrm{Tr}(YY^\dagger)]^3$ vertex in double line notation.

where the $1/N^4$ arises from the N^2 in front of the action and a $1/N^6$ from (4.1).

Note that the potential (4.2) is unbounded above and below, whatever the sign of f. An important question, raised in [5], is whether quantum corrections stabilize the potential. One can readily check that, unlike in the $D = 4$, $\mathcal{N} = 4$ SYM theory studied in [8] and used for cosmology in [6], the beta function for the coupling f vanishes to leading order in the $1/N$ expansion. However, at next to leading order we find corrections from the diagrams in Figs. 2 and 3.

It is important to know whether ABJM theory deformed by the triple trace potential (4.2) can be defined without a UV cutoff (and if a UV cutoff is necessary, whether it influences the dynamics of interest). Since a complete analysis appears rather complicated we shall, as in Ref. [5], begin by studying simpler but analogous scalar field models sharing key features with the theory of interest. In fact, in the limit of weak 't Hooft coupling, $N/k \to 0$, our deformation of ABJM theory precisely reduces to the $O(2N^2) \times O(2N^2)$ vector model at large N. In Sec. IV A, we first discuss the $O(N)$ vector model, drawing an important distinction between the cases with positive and negative coupling. In Sec. IV B, we then study the $O(N) \times O(N)$ vector model, which appears as the weak 't Hooft coupling limit of the deformed ABJM model of

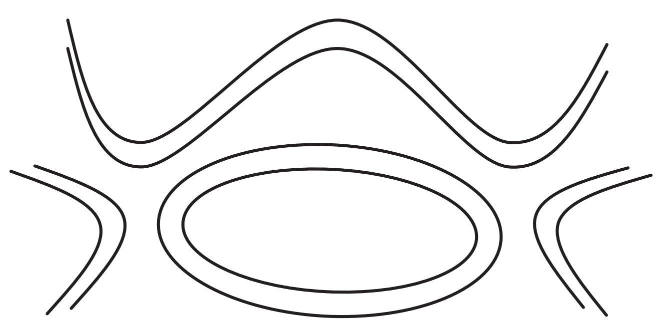

FIG. 2. Two-loop diagram that renormalizes the coupling f at order $1/N^2$.

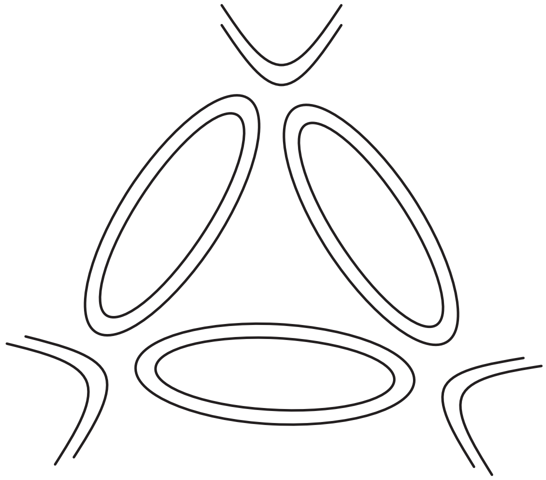

FIG. 3. Four-loop diagram that renormalizes the coupling f at order $1/N^2$.

interest. Finally, in Sec. IV C, we comment on the case of nonzero 't Hooft coupling.

A. The $O(N)$ vector model

Before discussing the subleading corrections in the model of interest, let us first discuss the analogous question in the well-understood $O(N)$ vector model at the tricritical point. The latter model, which describes N scalar fields in three dimensions, assembled in a vector $\vec{\phi}$, is defined by the action

$$S = \int d^3x\left(-\frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} - \frac{1}{6}\frac{\lambda}{N^2}(\vec{\phi}\cdot\vec{\phi})^3\right). \tag{4.3}$$

The sextic vertex is the analogue of Fig. 1, with all double lines replaced by single lines. The perturbative beta function for λ vanishes to leading order in the $1/N$ expansion, but receives nonzero contributions of order λ^2/N and λ^3/N from the logarithmically divergent two- and four-loop diagram analogous to Figs. 2 and 3 (with all double lines replaced by single lines). (Contributions with higher powers of λ are suppressed by additional powers of $1/N$.) The sum of the Feynman diagrams in Figs. 1–3 is [21–23]

$$3!2^3\left[-i\frac{\lambda}{6N^2} + \frac{i}{2}\ln\left(\frac{\Lambda^2}{p^2}\right)\frac{\lambda^2}{36N^4}\frac{9N}{\pi^2} - \frac{i}{2}\ln\left(\frac{\Lambda^2}{p^2}\right)\frac{\lambda^3}{216N^6}\frac{9N^3}{32\pi^2}\right], \tag{4.4}$$

times the appropriate tensor, where Λ is a UV cutoff and the scale p^2 is set by (spacelike) momenta flowing in and out of the diagrams. From (4.4), one reads off the beta function

$$\beta(\lambda) = \frac{3}{2\pi^2 N}\left(\lambda^2 - \frac{\lambda^3}{192}\right), \tag{4.5}$$

which is indeed suppressed by $1/N$.

For a positive potential ($\lambda > 0$), it follows from the quadratic term in (4.5) that the coupling is marginally irrelevant for small values of λ, for which it increases towards the UV. As λ increases, the cubic term in (4.5) becomes important and a perturbative UV fixed point is reached at

$$\lambda^* = 192. \tag{4.6}$$

In [12], a self-consistent, static, nonperturbative UV fixed point was found, in the strict $N = \infty$ limit, at the smaller value

$$\lambda_c = 16\pi^2 < \lambda^*, \tag{4.7}$$

and an instability was established for $\lambda > \lambda_c$, meaning that if one attempts to construct a static vacuum, all masses are of the order of the cutoff, so that the theory does not possess a continuum limit; see [5] for a recent discussion. However, preliminary results indicate that there is no such UV dependence in the time-dependent backgrounds of interest to us [14].

For a negative potential ($\lambda < 0$), the quadratic term in (4.5) implies that the coupling is asymptotically free (as discussed in [10] for $-\phi^4$ theory in four dimensions). As mentioned in [10] (see Appendix B of [6] for a recent discussion in a context closely related to the present paper, and the discussion below), one can then use the techniques of [24] to show directly that the energy of the system is unbounded below.[4] So at least for the $O(N)$ vector model, the fact that the potential with $\lambda < 0$ is unbounded below definitely survives quantum corrections.[5]

We now compute the Coleman-Weinberg effective potential in the regime $-1 \ll \lambda < 0$, so that the first term in (4.5) dominates. The coupling as a function of the renormalization scale μ is determined by the Callan-Symanzik equation

$$\mu\frac{d\lambda}{d\mu} = \frac{3\lambda^2}{2\pi^2 N}, \tag{4.8}$$

whose solution is

$$\lambda_\mu = -\frac{4\pi^2 N}{3\ln(\mu^2/M^2)}, \tag{4.9}$$

with M being an arbitrary mass scale (implementing dimensional transmutation). The Coleman-Weinberg potential, where the renormalization scale is set by a field value, is then

[4]Based on this, the authors of [10] dismissed the theory as nonsense, whereas in [6] a first attempt was made to make sense of such field theories. An update on the latter work will appear elsewhere [14].

[5]This conclusion might at first sight appear different from that reached in [5]. However, looking more closely, the computations referred to in [5] refer to *positive* coupling (beyond the critical one).

$$V(\vec{\phi}) = -\frac{4\pi^2}{3N\ln[(\vec{\phi}\cdot\vec{\phi})/M^2]}(\vec{\phi}\cdot\vec{\phi})^3. \tag{4.10}$$

If M is chosen such that the coupling λ is small, $-1 \ll \lambda < 0$, for some value of $\vec{\phi}\cdot\vec{\phi}$, it will be even smaller for larger values. [Note that the condition $-1 \ll \lambda < 0$ implies that the logarithm in (4.10) should be at least of order N.] Therefore, the perturbative analysis leading to the potential (4.10) is reliable for sufficiently large field values, which establishes that it is unbounded below.

B. The $O(N)\times O(N)$ vector model

Consider the $O(N)\times O(N)$ vector model defined by the action

$$S = \int d^3x\Big[-\frac{1}{2}\partial_\mu\vec{\phi}_1\cdot\partial^\mu\vec{\phi}_1 - \frac{1}{2}\partial_\mu\vec{\phi}_2\cdot\partial^\mu\vec{\phi}_2 - \frac{\lambda_{111}}{6N^2}(\vec{\phi}_1\cdot\vec{\phi}_1)^3 - \frac{\lambda_{222}}{6N^2}(\vec{\phi}_2\cdot\vec{\phi}_2)^3 - \frac{\lambda_{112}}{6N^2}(\vec{\phi}_1\cdot\vec{\phi}_1)^2 \times(\vec{\phi}_2\cdot\vec{\phi}_2) - \frac{\lambda_{122}}{6N^2}(\vec{\phi}_1\cdot\vec{\phi}_1)(\vec{\phi}_2\cdot\vec{\phi}_2)^2\Big]. \tag{4.11}$$

For the special case $\lambda_{112} = -\lambda_{122} = -3\lambda_{111} = 3\lambda_{222} \equiv -3\lambda$, this corresponds to the potential

$$V = \frac{\lambda}{6N^2}(\vec{\phi}_1\cdot\vec{\phi}_1 - \vec{\phi}_2\cdot\vec{\phi}_2)^3. \tag{4.12}$$

By collecting the $2N^2$ real components of the complex $N\times N$ matrix Y^1 in a $2N^2$-component vector $\vec{\phi}_1$, and similarly for Y^2, we see that the triple trace potential (4.2) takes the form (4.12) (with N replaced by $2N^2$). Moreover, in the $N/k\to 0$ weak 't Hooft coupling limit, the deformed ABJM action precisely reduces to that of the $O(2N^2)\times O(2N^2)$ vector model.

When we consider the potential (4.12), we see that there are four terms, and that the potential does not have a definite sign. In fact, even if they appear in fixed ratios in the classical potential (4.12), the couplings of the four terms will renormalize differently, which is why we include four different couplings in (4.11) to investigate the ultraviolet properties of the theory with potential (4.12).[6]

It is straightforward to generalize the perturbative beta function (4.5) at order $1/N$ to the model (4.11). The result is

$$\frac{2\pi^2N}{3}\beta_{111} = \lambda_{111}^2 + \frac{1}{9}\lambda_{112}^2 - \frac{1}{192}\Big(\lambda_{111}^3 + \frac{1}{3}\lambda_{111}\lambda_{112}^2 + \frac{1}{9}\lambda_{112}^2\lambda_{122} + \frac{1}{27}\lambda_{122}^3\Big);$$

$$\frac{2\pi^2N}{3}\beta_{112} = \frac{1}{9}\lambda_{112}^2 + \frac{1}{9}\lambda_{122}^2 + \frac{2}{3}\lambda_{111}\lambda_{112} + \frac{2}{9}\lambda_{112}\lambda_{122} - \frac{1}{192}\Big(\lambda_{111}^2\lambda_{112} + \frac{2}{3}\lambda_{111}\lambda_{112}\lambda_{122} + \frac{1}{9}\lambda_{112}^3 + \frac{1}{3}\lambda_{112}^2\lambda_{222} + \frac{2}{9}\lambda_{112}\lambda_{122}^2 + \frac{1}{3}\lambda_{122}^2\lambda_{222}\Big);$$

$$\frac{2\pi^2N}{3}\beta_{122} = \frac{1}{9}\lambda_{122}^2 + \frac{1}{9}\lambda_{112}^2 + \frac{2}{3}\lambda_{222}\lambda_{122} + \frac{2}{9}\lambda_{122}\lambda_{112} - \frac{1}{192}\Big(\lambda_{222}^2\lambda_{122} + \frac{2}{3}\lambda_{222}\lambda_{122}\lambda_{112} + \frac{1}{9}\lambda_{122}^3 + \frac{1}{3}\lambda_{122}^2\lambda_{111} + \frac{2}{9}\lambda_{122}\lambda_{112}^2 + \frac{1}{3}\lambda_{112}^2\lambda_{111}\Big);$$

$$\frac{2\pi^2N}{3}\beta_{222} = \lambda_{222}^2 + \frac{1}{9}\lambda_{122}^2 - \frac{1}{192}\Big(\lambda_{222}^3 + \frac{1}{3}\lambda_{222}\lambda_{122}^2 + \frac{1}{9}\lambda_{122}^2\lambda_{112} + \frac{1}{27}\lambda_{112}^3\Big). \tag{4.13}$$

From our knowledge of the $O(N)$ vector model, reviewed above, we can immediately infer the existence of the following perturbative fixed points. One fixed point simply corresponds to the nontrivial UV fixed point of the $O(2N)$ vector model: $\lambda_{112} = \lambda_{122} = 3\lambda_{111} = 3\lambda_{222} = 3\lambda^*$. A second fixed point has all couplings equal to zero; it can be approached in the UV by starting with the $O(2N)$ vector model with small negative coupling. The perturbative fixed point of interest to us has $\lambda_{222} = \lambda^*$ and $\lambda_{112} = \lambda_{122} = \lambda_{111} = 0$. By integrating (4.13) numerically (see Fig. 4), we find that this UV fixed point is reached when we start with (4.12) and flow towards the UV. If we choose conventions such that $\lambda < 0$ in (4.12) (the other sign is related to this by interchanging $\vec{\phi}_1$ and $\vec{\phi}_2$), λ_{111} remains negative and approaches zero as the renormalization scale is increased, just as we encountered in the $O(N)$ vector model with negative coupling. From (4.13), one determines

[6]Incidentally, at large 't Hooft coupling, one would use the bulk description to investigate the renormalization properties of the triple trace potential (4.2). Since the field φ dual to the operator $\mathcal{O}$ is part of the consistent truncation (2.1), it does not source the other scalar fields and one might be tempted to conclude that at large 't Hooft coupling, unlike what happens at weak 't Hooft coupling, the potential (4.2) preserves its form (4.12) under renormalization group flow. However, as in [25], pure AdS continues to be a solution to the classical equations of motion of (2.1) even with modified boundary conditions (2.4), which indicates that there is no running at the level of classical supergravity. (Moreover, unlike the double trace deformation analyzed in [25], the modified boundary conditions preserve the asymptotic AdS symmetry group.) It therefore seems to us that seeing any nontrivial renormalization group flow will require going beyond the classical supergravity approximation, and for this it is important to know whether properties related to consistent truncation survive quantum corrections in the bulk.

the detailed UV-limiting behavior of the four couplings,

$$\lambda_{111} \to \frac{1}{27(192)^2}\lambda_{122}^3, \qquad \lambda_{112} \to \frac{1}{576}\lambda_{122}^2,$$
$$\lambda_{122} \to C\mu^{-(96/\pi^2 N)}, \qquad \lambda_{222} \to 192, \qquad \mu \to \infty, \tag{4.14}$$

with C a negative constant (see Fig. 4). This behavior will be important in our discussion of the related cosmology [14].

As we have already mentioned, it is known that, at large N, nonperturbative effects destabilize the model for positive coupling $\lambda > \lambda_c$, so that it does not possess a UV-independent, static ground state [12,13]. However, for describing cosmology, and cosmological singularities, in particular, we are not interested in static ground states, and whether any UV-dependence enters depends on the questions being asked. We shall detail this point in future work [14], where we use the model presented here to study the cosmological space-times mentioned in the Introduction.

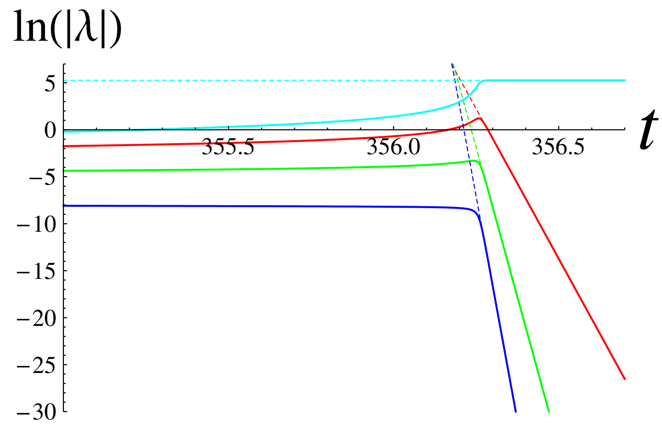

FIG. 4 (color online). Numerical solution for the running couplings in the $O(N) \times O(N)$ vector model, shown against $t \equiv 3\ln(\mu)/(2\pi^2 N)$. Initial conditions were specified by (4.12), with $\lambda = -1/500$, at $t = 0$. From top to bottom, the solid curves show λ_{222}, λ_{122}, λ_{112}, and λ_{111}. The dashed lines show the limiting behavior of the couplings in the UV, given in (4.14). Note that the absolute values of the couplings are shown: λ_{111} and λ_{122} are negative whereas λ_{112} and λ_{222} are positive.

C. Triple trace deformation of ABJM theory: Comments

In the previous subsection, we have studied the vector model arising in the limit of zero 't Hooft coupling, in which we could ignore the single trace interactions in the ABJM action (3.1). As in [6], the bulk is in a stringy regime for weak 't Hooft coupling. However, from the bulk analysis described in Sec. II, which is valid at large 't Hooft coupling, we know that at least certain important features, such as the unboundedness of the potential and the absence of logarithmic running to leading order in $1/N$, extend to the regime with large 't Hooft coupling. Another question one may ask is whether the beta functions of the deformation couplings will receive corrections linear in f, for instance proportional to f/k^2 rather than f^2. However, such logarithmically divergent diagrams need to cancel because $\mathcal{O}$ is a chiral primary operator, whose anomalous dimension must vanish in the undeformed theory: from any logarithmically divergent diagram with one triple trace and one single trace vertex contributing to the triple trace coupling, one can construct a logarithmically divergent diagram contributing to the anomalous dimension of $\mathcal{O}$ by stripping off two uncontracted $\mathrm{Tr}(YY^\dagger)$ factors from the triple trace vertex.

An important difference with the $D = 4$, $\mathcal{N} = 4$ SYM theory, where the running of f occurred at leading (zeroth) order in the $1/N$ expansion, is that here the running of f is suppressed by $1/N^2$. This is consistent with the absence of logarithmic terms in the asymptotic bulk supergravity solutions, mentioned at the end of Sec. II, and can therefore be regarded as a test of the AdS/CFT correspondence.

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Adiabaticity and emergence of classical space-time in time-dependent matrix theories

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Abstract: We discuss the low-curvature regime of time-dependent matrix theories proposed to describe non-perturbative quantum gravity in asymptotically plane-wave space-times. The emergence of near-classical space-time in this limit turns out to be closely linked to the adiabaticity of the matrix theory evolution. Supersymmetry restoration at low curvatures, which is crucial for the usual space-time interpretation of matrix theories, becomes an obvious feature of the adiabatic regime.

Keywords: M(atrix) Theories, Penrose limit and pp-wave background, M-Theory

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Contents

1 Introduction

Time-dependent matrix theories [1] have been introduced as an analogue of the Banks-Fischler-Shenker-Susskind flat space-time matrix theory [2] and of matrix string theory [3–5] in an attempt to describe non-perturbative quantum gravity in time-dependent, possibly highly curved (or even singular) space-times. The original set-up of [1] has been later extended in various directions [6–17]; in particular, a systematic generalization of the analysis to more general singular homogeneous plane-wave space-time backgrounds has appeared in [17]. In close parallel to the flat space matrix theory conjectures, one may expect these models to give a complete quantum-gravitational theory of asymptotically plane-wave space-times.

The usual construction of (time-independent) matrix theories [2] essentially relies on the type IIA superstring/M-theory duality conjecture. Namely, one compactifies a light-like dimension in the background 11-dimensional Minkowski space-time of M-theory, i.e., performs a discrete light-cone quantization (DLCQ) [18]. To make the construction more precise [19] (see also [20]), one takes this compact dimension to be slighly space-like. A large boost relates this theory to M-theory with a manifestly space-like compactification on a very small circle. The latter system is related to weakly coupled type IIA string theory via the type IIA superstring/M-theory duality conjecture. Furthermore, for N units of momentum on the DLCQ circle and finite energies in the original reference frame used for DLCQ, N D0-branes have to be present in the type IIA theory, and the only surviving degrees of freedom are the massless open string states associated with the D0-branes. This set-up yields the matrix theory action. An analogue of this argument can be devised for the case of IIA superstring theory (rather than M-theory), yielding a matrix string action [3–5] (rather than matrix quantum mechanics).

Because plane waves enjoy a light-like isometry, the Minkowski space arguments can be generalized to plane-wave backgrounds, resulting in time-dependent matrix theories and matrix string theories. Even though, in the context of these models, novel physics is expected to emerge in the high-curvature regions of the time-dependent backgrounds, it is also important to understand in what precise manner the dynamics approaches classical space-time when the curvatures become small.

A related issue is supersymmetry restoration. Supersymmetry is essential in the flat space matrix theory to protect the free propagation of gravitons, which in turn underlies the conventional space-time interpretation. In time-dependent matrix theories, supersymmetry is broken completely (whichever part is not broken by the plane wave backround, will be broken by the light-like momentum on the DLCQ circle). It is therefore crucial to understand why this does not prevent space-time from forming (at least in the low-curvature regime), an issue that has been raised since the original formulation of this class of models in [1].

Heuristically, the question of low-curvature dynamics has been addressed already in [1]: it has been shown that, in an appropriate parametrization, the time-dependent matrix string theory reduces to a 2-dimensional supersymmetric gauge theory on a Milne space-time with metric

$$ds^2 = e^{2\eta}\left(-d\eta^2 + dx^2\right), \qquad x \sim x + 2\pi. \tag{1.1}$$

The supersymmetry is only broken by the identification $x \sim x + 2\pi$, and, since the radius of the Milne circle becomes large at late times (i.e., in the low curvature regime), one could expect that, for a wide range of processes, the supersymmetry breaking will become invisible. In [21], the effective potential for the matrix string variables has been computed in the weak coupling expansion of the gauge theory. Even though the computation is not technically valid at late times, formal extrapolation of the resulting expressions suggested that the effective potential decays at late times, a feature that could be indicative of supersymmetry restoration. Another attempt to study late time (low background curvature) dynamics of the time-dependent matrix theories has been recently undertaken in [22].

Our present objective is to re-address these issues in a maximally clear and simple fashion. For the 11-dimensional case of [6], the matrix action previously presented in the literature shows steep time dependences at late times. This makes the emergence of a near-classical space-time quite puzzling, as it superficially suggests a strong explicit supersymmetry breaking, among other things. We shall show that, treated in appropriate variables, the relevant time-dependent matrix theory approaches its flat space counterpart at late times (low curvatures). The manner of convergence is somewhat subtle, but, since a bound on deviations from the flat space theories will be given, the issues of space-time interpretation and supersymmetry restoration are automatically resolved. One can understand this situation in a different way: the superficially steep time dependences in the original matrix theory action actually enter an adiabatic[1] regime at late times, which again connects the dynamics to that of a time-independent matrix theory. For the 11-dimensional case, this is simply an equivalent and less straightforward way to view the geometry-inspired variable redefinition that eliminates the time-dependences at late times.

[1]Adiabaticity has recently surfaced [23] in the context of quantum gravity in time-dependent backgrounds, though the precise setting differs substantially from ours.

For the 10-dimensional case of [1, 17], however, there appears to be no canonical variable redefinition that eliminates the time dependence in the Lagrangian at late times. (For instance, the 10-dimensional theory of [1] can be seen as the 11-dimensional matrix theory described in the previous passage, with an additional compactification. However, if we attempt to perform the same variable redefinition as in the 11-dimensional case, the compactification radius becomes time-dependent, making the theory awkward to study.) Still, one can establish the onset of an adiabatic regime in the late-time dynamics of these theories, which again connects them to their flat-space counterparts. (In justifying adiabaticity for this case, we shall rely on the standard flat-space matrix string theory conjectures [5], which are essential to a meaningful interpretation of the time-dependent matrix theories in any case, and hence already implicitly assumed.)

Note that in the present paper it is not our objective to trace the complete evolution of states from early to late times and show that a near-classical space-time emerges from a generic initial state (which we do not expect to be the case). Explaining the emergence of a near-classical space-time from initial conditions is a major problem in cosmology, which we do not address here. Rather, we shall show that the dynamics of the relevant matrix theories approaches their flat-space counterparts, if studied at late times. This implies that, given a state with a late-time near-classical space-time interpretation, matrix theory can consistently describe its further evolution. Establishing this property is already non-trivial, and it is an essential pre-requisite for a more thorough treatment of cosmological scenarios in our framework.

We shall start our exposition by analyzing the 11-dimensional time-dependent matrix theory, followed by the 10-dimensional matrix string theory. In these treatments, we shall perform the algebraic manipulations explicitly, leading up to the derivation of the necessary dynamical bounds. In the last section, we shall explain how our formal manipulations are related to quantum adiabatic theory.

2 Matrix quantum mechanics

We shall start by briefly reviewing the 11-dimensional (quantum-mechanical) matrix theories introduced in [6] as simpler analogues of the matrix string theories of [1]. The relevant 11-dimensional (M-theory) background has the form

$$ds^2 = e^{2\alpha x^+}\left(-2dx^+dx^- + (dx^i)^2\right) + e^{2\beta x^+}(dx^{11})^2, \tag{2.1}$$

or, in terms of the light-like geodesic affine parameter $\tau = e^{2\alpha x^+}/2\alpha$,

$$ds^2 = -2d\tau dx^- + 2\alpha\tau(dx^i)^2 + (2\alpha\tau)^{\beta/\alpha}(dx^{11})^2. \tag{2.2}$$

This metric satisfies the 11-dimensional supergravity equations of motion if the constants α and β are related as $\beta = -2\alpha$, or $\beta = 4\alpha$. The fact that these relations need to be imposed will not be relevant for our present considerations (it is essential, however, for the general consistency of the corresponding matrix theories).

Since translations in x^- form an isometry of the above background, the usual DLCQ argument (proposed in [19] and adapted to the time-dependent case in [1]) can be applied.

The result [6] is a matrix theory that can be expected to describe non-perturbative quantum gravity in space-times asymptotic to (2.2). The bosonic and fermionic parts of the matrix theory action, respectively, have the following form:

$$
\begin{aligned}
S_B &= \int d\tau \mathrm{Tr} \left\{ \frac{\alpha\tau}{R}(D_\tau X^i)^2 + \frac{(2\alpha\tau)^{\beta/\alpha}}{2R}(D_\tau X^{11})^2 - \frac{R}{4}(2\alpha\tau)^2 [X^i, X^j]^2 \right. \\
&\qquad\qquad \left. - \frac{R}{2}(2\alpha\tau)^{1+\beta/\alpha}[X^i, X^{11}]^2 \right\}, \\
S_F &= \int d\tau \left\{ i\theta^T D_\tau \theta - R\sqrt{2\alpha\tau}\theta^T \gamma_i [X^i, \theta] - R(2\alpha\tau)^{\beta/2\alpha}\theta^T \gamma_{11}[X^{11}, \theta] \right\}.
\end{aligned} \tag{2.3}
$$

The problems we intend to discuss can be understood already at the level of the action (2.3). The space-time backgrounds implicit in (2.3) are supposed to feature a light-like singularity at $\tau = 0$ and become progressively more classical at large τ. Yet, the explicit time dependences in (2.3) superficially become more steep, if anything, at large τ. This issue certainly deserves further clarification. Additionally, there are no supersymmetries explicit in (2.3). Since supersymmetries are crucial for the free propagation of well-separated gravitons (and hence, a robust geometrical interpretation) in the flat space matrix theory, one should attempt to find an analogue of supersymmetry in (2.3) that would enforce a similar type of dynamics.

To address these important issues, we first note that the metric (2.2) describes a plane wave and, with the coordinate transformation

$$
u = \tau, \quad z^i = \sqrt{2\alpha\tau}x^i, \quad z^{11} = (2\alpha\tau)^{\beta/2\alpha}x^{11}, \quad v = x^- + \frac{\alpha(z^i)^2 + \beta(z^{11})^2}{4\alpha\tau}, \tag{2.4}
$$

it can be brought to the Brinkmann form

$$
ds^2 = -2du\, dv - \frac{\alpha^2(z^i)^2 - (\beta^2 - 2\alpha\beta)(z^{11})^2}{(2\alpha u)^2}du^2 + (dz^i)^2 + (dz^{11})^2. \tag{2.5}
$$

This parametrization forces the metric to manifestly approach Minkowski space-time for large values of the light-cone time, which strongly suggests that the large time dynamics of the corresponding matrix theory will likewise approach the flat space matrix theory, if treated in appropriate variables. (As we shall see below, the convergence towards this limit is somewhat subtle, but the naïve expectation will prove well-grounded.)

The matrix theory corresponding to (2.5) can be constructed from 'scratch', as the metric enjoys the same v-translation isometry as (2.2). More straightforwardly, one can apply the matrix analogue[2] of the transformations (2.4) directly to (2.3) to obtain

$$
\begin{aligned}
S_B &= \int d\tau \mathrm{Tr} \left\{ \frac{1}{2R}\left[(D_\tau Z^i)^2 + (D_\tau Z^{11})^2\right] - \frac{R}{4}\left[[Z^i, Z^j]^2 + 2[Z^i, Z^{11}]^2\right] \right. \\
&\qquad\qquad \left. - \frac{\alpha^2(Z^i)^2 - (\beta^2 - 2\alpha\beta)(Z^{11})^2}{(2\alpha\tau)^2} \right\}, \\
S_F &= \int d\tau \left\{ i\theta^T D_\tau\theta - R\theta^T\gamma_i[Z^i, \theta] - R\theta^T\gamma_{11}[Z^{11}, \theta] \right\}.
\end{aligned} \tag{2.6}
$$

[2] Note that there is no matrix variable corresponding to v, since v becomes the DLCQ circle in the standard formulation of the matrix theory. The remaining transformations are linear, so the issue of matrix multiplication ordering never arises.

Note that the Brinkmann form of the matrix action has previously appeared [17] in the literature (for the 10-dimensional case). However, the action in [17] corresponds to bringing the 10-dimensional metric to the Brinkmann form, not the 11-dimensional metric (as we have done presently). Both metrics are of the plane-wave form.

The action (2.6) only differs from the flat space matrix theory by a term decaying as $1/\tau^2$, thus one may expect that the late time dynamics will be approximated by the flat space matrix theory and admit the usual space-time interpretation. However, the decay is quite slow and one might be worried about whether it is sufficient to ensure convergence.

To illustrate these worries, one may look at the straightforward example of a harmonic oscillator whose frequency depends on time as $1/t^2$:

$$\ddot{x} + \frac{k}{t^2}x = 0. \tag{2.7}$$

The two independent solutions to this equation can be given as t^a and t^{1-a}, where a is a k-dependent number. These two solutions are obviously quite different from a free particle trajectory, even though the equation of motion approaches that of a free particle at late times. The reason for this discrepancy is the slow rate of decay of the second term in (2.7).

However, in a physical setting, one is only able to perform *finite time* experiments. That is, one has to specify the initial values $x(t_0) = x_0$, $\dot{x}(t_0) = v_0$ and examine the corresponding solution between t_0 and $t_0 + T$. The solution is given by

$$x(t) = \frac{x_0(1-a) - v_0 t_0}{1-2a}\left(\frac{t}{t_0}\right)^a + \frac{v_0 t_0 - x_0 a}{1-2a}\left(\frac{t}{t_0}\right)^{1-a}. \tag{2.8}$$

One can then see that $x(t_0 + T) = x_0 + v_0 T + O(T/t_0)$, i.e., it is approximated by a free motion arbitrarily well if the experiment starts sufficiently late.

It may be legitimately expected that the *finite time* behavior of the full time-dependent matrix theory given by (2.6) will be approximated arbitrarily well by the flat space matrix theory at late times, just as in the above harmonic oscillator example. We shall now prove it by constructing an elementary bound on dynamical deviations due to a small time-dependent term in the Schrödinger equation.

We start with the following Schrödinger equation:

$$i\frac{d}{dt}|\Phi\rangle = (H_0 + f(t)H_1)|\Phi\rangle\,, \tag{2.9}$$

where H_0 and H_1 are time-independent, and rewrite it in the interaction picture (with respect to H_0):

$$|\Phi\rangle = e^{-iH_0(t-t_0)}|\xi\rangle\,, \qquad i\frac{d}{dt}|\xi\rangle = f(t)e^{iH_0(t-t_0)}H_1 e^{-iH_0(t-t_0)}|\xi\rangle\,. \tag{2.10}$$

We then proceed to consider

$$\begin{aligned}\frac{d}{dt}\Big|\,|\xi(t)\rangle - |\xi(t_0)\rangle\Big|^2 &= -\frac{d}{dt}\left(\langle\xi(t_0)\,|\xi(t)\rangle + c.c.\right)\\ &= -if(t)\left(\langle\xi(t_0)|\,e^{iH_0(t-t_0)}H_1 e^{-iH_0(t-t_0)}|\xi(t)\rangle - c.c.\right).\end{aligned} \tag{2.11}$$

Integrating this expression between t_0 and t_0+T and making use of standard inequalities for absolute values and scalar products, we obtain:

$$\begin{aligned}
\Big|\,|\xi(t_0{+}T)\rangle - |\xi(t_0)\rangle\,\Big|^2 &= -i\int\limits_{t_0}^{t_0+T} dt f(t)\left(\left\langle e^{iH_0(t-t_0)}H_1e^{-iH_0(t-t_0)}\xi(t_0)\;|\xi(t)\rangle - c.c.\right)\right.\\
&\leq 2\int\limits_{t_0}^{t_0+T} dt|f(t)|\sqrt{\left(\left|e^{iH_0(t-t_0)}H_1e^{-iH_0(t-t_0)}\xi(t_0)\rangle\right)\right)^2}\sqrt{(|\xi(t)\rangle)^2}\\
&\leq 2\left(\max_{[t_0,t_0+T]}|f(t)|\right)\int\limits_0^T dt\sqrt{\langle\xi(t_0)|\,e^{iH_0t}H_1^2e^{-iH_0t}\,|\xi(t_0)\rangle}. \qquad (2.12)
\end{aligned}$$

Now, assume that $f(t)$ approaches 0 at large times and consider a fixed $|\xi(t_0)\rangle \equiv |\xi_0\rangle$ (so we consider the evolution with fixed duration T of the same initial state $|\xi_0\rangle$ starting at different initial times t_0). In this case, the first factor in the last line becomes arbitrarily small for large t_0, whereas the second factor does not depend on t_0. We then conclude that, for sufficiently late times, the finite time evolution of the state vector will be approximated arbitrarily well by $|\xi(t)\rangle = \text{const}$, i.e., by the evolution with $f(t)$ set identically to 0.

It is then a simple corollary of the above bound that the time-dependent matrix theory dynamics becomes approximated arbitrarily well at late times by the flat space matrix theory, and, in particular, the supersymmetry is asymptotically restored (with all the usual consequences, such as protection of the flat directions of the commutator potential, and free graviton propagation).

A note may be in order: even though (2.12) does show that for any given experiment (fixed $|\xi(t_0)\rangle \equiv |\xi_0\rangle$ and fixed T) deviations from the flat space matrix theory become arbitrarily small for sufficiently large t_0, this by no means implies that the convergence is uniform with respect to $|\xi_0\rangle$, which affects the value of the second factor in the last line of (2.12). This is as it should be: for any t_0 there will be some experiments that will be able to discriminate between the time-dependent and flat cases (with a given precision), yet, such experiments will have to become more and more specialized (and less and less possible) at late times.

3 Matrix string theory

The 10-dimensional "Matrix Big Bang" matrix string theories [1, 17] essentially differ from the 11-dimensional case we have considered above by the compactness of one of the space-time dimensions. This feature precludes a straighforward variable redefinition that would reduce the Lagrangian to a sum of time-independent terms and terms decaying at large times (because the compactification radius would become time-dependent in the new variables). We shall therefore need to resort directly to adiabaticity-inspired arguments to establish the emergence of near-classical space-time far away from the Matrix Big Bang singularity.

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The aim of the construction of matrix string theories [3–5] is to develop a non-perturbative description of quantum gravity in 10 dimensions (with the perturbative limit of this description given by the usual perturbative type IIA string theory). For the original time-dependent Matrix Big Bang matrix string theory of [1], the 10-dimensional geometry is asymptotic to the linear dilaton configuration:

$$\begin{aligned} ds_{st}^2 &= -2dy^+dy^- + (dy^i)^2, \\ \phi &= -Qy^+. \end{aligned} \tag{3.1}$$

To construct the matrix string theory for the background (3.1), one first lifts the background (3.1) to 11 dimensions via the usual conjecture of type IIA/M-theory correspondence. The resulting 11-dimensional space-time is

$$ds^2 = e^{2Qy^+/3}\left(-2dy^+dy^- + (dy^i)^2\right) + e^{-4Qy^+/3}(dy)^2, \tag{3.2}$$

where y is a coordinate along the M-theory circle. This is followed by the DLCQ compactification of the light-like v-coordinate, interpreted as the M-theory circle of an "auxiliary" type IIA string theory. A T-duality [24] then relates the resulting theory of D0-branes on a compact dimension, i.e., a BFSS-like matrix theory with a compactified dimension, to a more manageable theory of wrapped D1-branes. This procedure has been carried out (in a slightly different but equivalent way) in [1] and has been reviewed in [25] and [17]. The resulting matrix string action is

$$S = \frac{1}{2\pi\ell_s^2}\int \mathrm{tr}\left(\frac{1}{2}(D_\mu X^i)^2 + \theta^T \not{D}\,\theta + \frac{1}{4g_{\mathrm{YM}}^2}F_{\mu\nu}^2 - g_{\mathrm{YM}}^2[X^i,X^j]^2 + g_{\mathrm{YM}}\theta^T\gamma_i[X^i,\theta]\right), \tag{3.3}$$

with the Yang-Mills coupling g_{YM} related to the worldsheet values of the dilaton:

$$g_{\mathrm{YM}} = \frac{\mathrm{e}^{-\phi(y^+(\tau))}}{2\pi l_s g_s} = \frac{e^{Q\tau}}{2\pi l_s g_s}. \tag{3.4}$$

A generalization of this set-up has been proposed [17]. One can start with a 10-dimensional power-law plane wave:

$$\begin{aligned} ds_{st}^2 &= -2dy^+dy^- + g_{ij}(y^+)dy^i dy^j \equiv -2dy^+dy^- + \sum_i (y^+)^{2m_i}(dy^i)^2 \\ &= -2dz^+dz^- + \sum_a \frac{m_a(m_a-1)}{(z^+)^2}(z^a)^2(dz^+)^2 + \sum_a (dz^a)^2, \\ \mathrm{e}^{2\phi} &= (y^+)^{3b/(b+1)} = (z^+)^{3b/(b+1)}. \end{aligned} \tag{3.5}$$

Here, the first and the second line represent the Rosen and the Brinkmann form of the same plane wave, respectively. In order for the supergravity equations of motion to be satisfied, one needs to impose [17]

$$\sum_i m_i(m_i-1) = -\frac{3b}{b+1}. \tag{3.6}$$

The original background of [1] can be seen as a $b \to -1$ limit of the above space-time [17].

The 11-dimensional space-time corresponding to (3.5) is

$$
\begin{aligned}
ds_{11}^2 &= -2dudv + \sum_i u^{2n_i}(dy^i)^2 + u^{2b}(dy)^2 \\
&= -2dudw + \sum_a \frac{n_a(n_a-1)}{u^2}(x^a)^2(du)^2 + \frac{b(b-1)}{u^2}x^2(du)^2 + \sum_a (dx^a)^2 + (dx)^2,
\end{aligned} \tag{3.7}
$$

with n_i related to m_i by $2m_i = (2n_i + b)/(b+1)$. Note that, since the Rosen and Brinkmann coordinates (first and second line in the above formula) are related by a u-dependent rescaling of the transverse coordinates, the identification of the (compact) x-variable is u-dependent. Thus, even though the second line of (3.7) approaches a flat space-time at large values of u, the time-dependent identification of x makes an immediate application of the derivations of the previous section impossible.

The usual formulation leads to a matrix string action, whose bosonic part is given, in the Rosen coordinates of (3.5), by

$$
\begin{aligned}
S_{\rm RC} = \int d\tau d\sigma \,\mathrm{Tr}\Big(&-\frac{1}{4}g_{\rm YM}^{-2}\eta^{\alpha\gamma}\eta^{\beta\delta}F_{\alpha\beta}F_{\gamma\delta} - \frac{1}{2}\eta^{\alpha\beta}g_{ij}(\tau)D_\alpha X^i D_\beta X^j \\
&+\frac{1}{4}g_{\rm YM}^2 g_{ik}(\tau)g_{jl}(\tau)[X^i, X^j][X^k, X^l]\Big).
\end{aligned} \tag{3.8}
$$

with the transverse metric g_{ij} given by the first line of (3.5) and the Yang-Mills coupling by

$$
g_{\rm YM} = \frac{\mathrm{e}^{-\phi(y^+(\tau))}}{2\pi l_s g_s} = \frac{\tau^{-3b/2(b+1)}}{2\pi l_s g_s}. \tag{3.9}
$$

One can further transform this action to the Brinkmann coordinates of the original plane wave, given by the second line of (3.5), to obtain [17]:

$$
\begin{aligned}
S_{\rm BC} = \int d\tau d\sigma \,\mathrm{Tr}\Big(&-\frac{1}{4}g_{\rm YM}^{-2}F_{\tau\sigma}^2 - \frac{1}{2}\left(D_\tau Z^a D_\tau Z^a - D_\sigma Z^a D_\sigma Z^a\right) \\
&+\frac{1}{4}g_{\rm YM}^2[Z^a, Z^b][Z^a, Z^b] + \frac{1}{2}A_{ab}(\tau)Z^a Z^b\Big),
\end{aligned} \tag{3.10}
$$

where $A_{ab} = \mathrm{diag}\{m_a(m_a-1)\}/\tau^2$. The latter form of the action only differs from a SYM gauge theory with a time-dependent coupling by the term involving A_{ab}. The contribution of this term at late times can be shown to be negligible by an argument very similar to the one employed for the 11-dimensional case in the previous section.

We thus end up at late times, both for the original Matrix Big Bang of [1] and for its generalization [17], with a super-Yang-Mills theory with a growing time-dependent coupling that is a power-law or exponential function of time, and we have to analyze this particular large time large coupling limit. Again, a puzzling feature here is that the time dependences remain steep at large values of τ (set equal to y^+ by the gauge choice), far away from the singularity of the original plane wave. The question of what happens at late times has been discussed in the literature [1, 22] without definitive quantitative answers provided.

The large coupling limit of the *time-independent* super-Yang-Mills theory is essential to the gravitational interpretation of the flat space matrix string theory [5]. Note that this limit is distinct from the one we have to consider (time-dependent versus time-independent theory), however, in the adiabatic regime that our considerations will establish, properties of the time-dependent theory are related to properties of time-independent "snapshots" of the time-dependent Hamiltonian. We shall now assume the standard conjectures about the large coupling behavior of the time-independent theory [5] and show that they result in an emergence of an adiabatic regime within the time-dependent theory. The relevant manipulations will be carried out in a pragmatic fashion, leading to a construction of the low-curvature limit. We shall explain the relation of these manipulations to the general adiabatic theory in the next section.

In [5], time-independent super-Yang-Mills theories have been conjectured to converge in the infrared ($g_{\rm YM} \to \infty$) limit to a sigma model conformal field theory on a permutation orbifold (with target space $S^N\mathrm{R}^8 \equiv (\mathrm{R}^8)^N/S_N$, where S_N permutes the N copies of R^8) at finite N (and to second quantized free strings on Minkowski space in the large N limit). The considerations given in support of this claim are not rigorous, so we shall have to make natural assumptions below about which particular form of convergence is implied.

Firstly, we assume that convergence occurs at the level of eigenstates: we postulate that there exists a family of eigenvectors $|\Psi_n(g_{\rm YM})\rangle$ of $H_{\rm YM}(g_{\rm YM})$ such that

$$\langle\{A,\theta(x)\}|\Psi_n(g_{\rm YM})\rangle = \Psi_n^{\rm perm.orb.} + \sum_{k=1}^{\infty}\frac{\delta\Psi_{nk}}{g_{\rm YM}^k}, \tag{3.11}$$

$$H_{\rm YM}\,|\Psi_n(g_{\rm YM})\rangle = \left(E_n^{\rm perm.orb.} + \sum_{k=1}^{\infty}\frac{\delta E_{nk}}{g_{\rm YM}^k}\right)|\Psi_n(g_{\rm YM})\rangle \equiv E_n(g_{\rm YM})\,|\Psi_n(g_{\rm YM})\rangle\,. \tag{3.12}$$

Here $|\{A,\theta(x)\}\rangle$ symbolically denotes eigenstates of the canonical coordinates of the super-Yang-Mills theory (the details depend on the quantization procedure). Their appearance in the above expressions is necessary to compare wave-vectors of super-Yang-Mills theories at different values of the coupling (which are technically defined in different Hilbert spaces). Further, $\Psi_n^{\rm perm.orb.}$ and $E_n^{\rm perm.orb.}$ denote wave-functionals of energy eigenstates of the permutation orbifold conformal field theory and their corresponding energies. (The above set-up assumes that the spectrum at finite $g_{\rm YM}$ is a continuous deformation of the limiting spectrum at $g_{\rm YM} \to \infty$. This may not always be so, for example, a discrete spectrum of bound states at finite $g_{\rm YM}$ may merge in the continuum at $g_{\rm YM} \to \infty$. However, we are not aware of any such states in the context of perturbative string theory, nor do we anticipate that they could have any dramatic effect on our considerations. We shall therefore assume (3.12) for our present purposes.) Note also that our summation over n is a symbolic notation that implies integration over the continuous spectrum and summation over all discrete eigenvector labels, and none of our derivations use any assumptions about discreteness of the spectrum.

If the Hilbert space were finite-dimensional, the above conditions would guarantee that the dynamical behavior of the theory reaches the free string limit suggested in [5] at large values of $g_{\rm YM}$. However, for an infinite-dimensional space of states, additional conditions enforcing not-too-poor convergence of highly excited eigenstates need to be imposed. Indeed, in a physical setting, in order to declare that one theory approaches

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another in a certain limit, one needs to ascertain that the evolutions of the two theories become arbitrarily close to each other for any finite energy normalizable initial state and for any finite duration of the experiment. Furthermore, this convergence needs to be uniform for all normalizable initial states with energies below some fixed value and all experiments lasting less than some given duration (for, were it not so, there would always exist some finite energy finite duration experiments discriminating between the two theories with a finite precision). We embody these conditions in a statement that

$$\sum_n c_n e^{-iE_n(g_{\rm YM})T}\,\langle\{A,\theta(x)\}|\Psi_n(g_{\rm YM})\rangle = \sum_n c_n e^{-iE_n^{\rm perm.orb.}T}\Psi_n^{\rm perm.orb.} + \sum_{k=1}^{\infty}\frac{\psi_k}{g_{\rm YM}^k}, \tag{3.13}$$

with $\sum|c_n|^2=1$ and ψ_k being a set of normalizable wave-functionals. We further impose that the norms of ψ_k are uniformly bounded for any T less than a chosen fixed value and for any initial state energy $\sum E_n^{\rm perm.orb.}|c_n|^2$ less than a chosen fixed value.

We shall assume something slightly stronger than (3.13). Namely, we shall assume that the Yang-Mills coupling constant in the energies and eigenvectors on the left-hand side of (3.13) can be sent to infinity independently, still resulting in a power series expansion:

$$\sum_n c_n e^{-iE_n(g_1)T}\,\langle\{A,\theta(x)\}|\Psi_n(g_2)\rangle = \sum_n c_n e^{-iE_n^{\rm perm.orb.}T}\Psi_n^{\rm perm.orb.} + \sum_{\substack{k,l=0\\(k,l)\neq(0,0)}}^{\infty}\frac{\psi_{kl}}{g_1^k g_2^l}, \tag{3.14}$$

with the same type of uniformity specifications we made under (3.13). We shall further assume that (3.14) can be differentiated with respect to T without losing uniformity, which yields (with T set to 0 and g_2 sent to infinity):

$$\sum_n c_n(E_n(g_{\rm YM}) - E_n^{\rm perm.orb.})\Psi_n^{\rm perm.orb.} = \sum_{k=1}^{\infty}\frac{\phi_k}{g_{\rm YM}^k}. \tag{3.15}$$

As before, all these conditions are identical to (3.12) for any finite-dimensional space of states, but in an infinite-dimensional space, they constrain the convergence (in $g_{\rm YM}$) at large n. (These conditions are still considerably milder than demanding uniformity of convergence in n, which would be quite unphysical.) There is a possibility that the derivations below can be made without relying on anything beyond (3.13), but we have not been able to devise such an argument.

Now we turn to the Schrödinger equation of the time-dependent super-Yang-Mills theory

$$i\frac{d}{dt}\,|\Psi\rangle = H_{\rm YM}(g_{\rm YM}(t))\,|\Psi\rangle \tag{3.16}$$

and expand the state vector in the basis given by (3.12):

$$|\Psi\rangle = \sum_n c_n(t)\,|\Psi_n(g_{\rm YM}(t))\rangle\,. \tag{3.17}$$

This yields

$$i\frac{dc_n}{dt} + i\sum_m c_m(t)\dot{g}_{\rm YM}\,\langle\Psi_n(g_{\rm YM})|\,\frac{d}{dg_{\rm YM}}\,|\Psi_m(g_{\rm YM})\rangle = E_n(g_{\rm YM})c_n(t). \tag{3.18}$$

We can rewrite this equation as

$$i\frac{dc_n}{dt} = E_n^{\text{perm.orb.}} c_n + \sum_m H_{nm} c_m(t), \tag{3.19}$$

where H_{nm} are bounded by a negative power of $g_{\rm YM}$:

$$H_{nm} = O(1/g_{\rm YM}^{\gamma}), \tag{3.20}$$

with the precise value of the power depending on whether $g_{\rm YM}$ is a power-law or exponential function of t. Note that the relevant coupling dependence coming from the second term on the left-hand side of (3.18) is $\dot{g}_{\rm YM}/g_{\rm YM}^2$, whereas the energy corrections on the right-hand side of (3.18) are of order $1/g_{\rm YM}$.

The fact that the matrix elements H_{mn} are decreasing functions of time suggests (though by no means in a conclusive way) that the evolution approaches that of the permutation orbifold CFT at late times. We shall now analyze (3.19) in more detail to establish that this convergence does indeed take place.

Consider first a Schrödinger equation of the form

$$i\frac{d}{dt}|\Phi\rangle = (H_0 + H_1(t))|\Phi\rangle, \tag{3.21}$$

where H_0 is time-independent.[3] In parallel to the derivations of the previous section, we can rewrite it in the interaction picture (with respect to H_0):

$$|\Phi\rangle = e^{-iH_0(t-t_0)}\,|\xi\rangle\,, \qquad i\frac{d}{dt}\,|\xi\rangle = e^{iH_0(t-t_0)}H_1(t)e^{-iH_0(t-t_0)}\,|\xi\rangle\,. \tag{3.22}$$

We then proceed to consider

$$\begin{aligned}\frac{d}{dt}\Big|\,|\xi(t)\rangle - |\xi(t_0)\rangle\,\Big|^2 &= -\frac{d}{dt}\left(\langle\xi(t_0)\,|\xi(t)\rangle + c.c.\right)\\ &= -i\left(\langle\xi(t_0)|\,e^{iH_0(t-t_0)}H_1(t)e^{-iH_0(t-t_0)}\,|\xi(t)\rangle - c.c.\right).\end{aligned} \tag{3.23}$$

Integrating this expression between t_0 and t_0+T and making use of standard inequalities for absolute values and scalar products, we obtain:

$$\begin{aligned}\Big|\,|\xi(t_0+T)\rangle - |\xi(t_0)\rangle\,\Big|^2 &= -i\int_{t_0}^{t_0+T} dt\,\Big(\Big\langle e^{iH_0(t-t_0)}H_1(t)e^{-iH_0(t-t_0)}\xi(t_0)\,|\xi(t)\rangle - c.c.\Big)\\ &\le 2\int_{t_0}^{t_0+T} dt\sqrt{\left\|e^{iH_0(t-t_0)}H_1(t)e^{-iH_0(t-t_0)}\,|\xi(t_0)\rangle\right\|^2}\sqrt{\|\,|\xi(t)\rangle\|^2}\\ &= 2\int_0^T dt\sqrt{\|H_1(t+t_0)e^{-iH_0 t}\,|\xi(t_0)\rangle\|^2}.\end{aligned} \tag{3.24}$$

[3] It is an interesting mathematical question, to which the authors do not know the answer, what precise conditions should be imposed on $H_1(t)$ in order to make its contribution to the evolution small. There are many notions of convergence for Hilbert space operators, and saying that $H_1(t)$ becomes small in some limit is vacuous, unless the precise manner of convergence is specified.

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For the Schrödinger equation (3.19), the state vector is given by the set of numbers $\{c_n\}$, and we take H_0 to be the first term on the right-hand-side, and $H_1(t)$ to be the second term (the detailed expression for H_1 can be read off from (3.18)). We furthermore denote $|\xi(t_0)\rangle = \{c_n^{(0)}\}$, and observe that

$$\begin{aligned} &H_1(t+t_0)e^{-iH_0t}\,|\xi(t_0)\rangle \\ &\quad\equiv \sum_m \left(\dot{g}_{\mathrm{YM}}\,\langle\Psi_n(g_{\mathrm{YM}}(t+t_0))|\,\frac{d}{dg_{\mathrm{YM}}}\,|\Psi_m(g_{\mathrm{YM}}(t+t_0))\rangle \exp\left[-iE_m^{\text{perm.orb.}}t\right] c_m^{(0)} \right) \\ &\qquad + (E_n(g_{\mathrm{YM}}(t+t_0)) - E_n^{\text{perm.orb.}}) \exp\left[-iE_n^{\text{perm.orb.}}t\right] c_n^{(0)}. \end{aligned} \tag{3.25}$$

The norm of the first term (i.e., the sum over n of the square of its absolute value) is equal to the norm of the Hilbert space vector

$$\dot{g}_{\mathrm{YM}} \sum_m c_m^{(0)} \exp\left[-iE_m^{\text{perm.orb.}}t\right] \frac{d}{dg_{\mathrm{YM}}}\,|\Psi_m(g_{\mathrm{YM}}(t+t_0))\rangle \tag{3.26}$$

(we use $\sum_n |\Psi_n(g_{\mathrm{YM}}(t+t_0))\rangle\,\langle\Psi_n(g_{\mathrm{YM}}(t+t_0))| = 1$) and it is bounded (uniformly with respect to t) by the g_2-derivative of the condition (3.14) with $g_1 \to \infty$ to be $O(\dot{g}_{\mathrm{YM}}/g_{\mathrm{YM}}^2)$. The norm of the second term is bounded (uniformly with respect to t) by (3.15) to be $O(1/g_{\mathrm{YM}})$. Since the bounds are uniform with respect to t, they can be immediately integrated in (3.24).

We then conclude that

$$\Big|\,|\xi(t_0+T)\rangle - |\xi(t_0)\rangle\,\Big|^2 = T\,O(1/g_{\mathrm{YM}}^{\gamma}(t_0)), \tag{3.27}$$

i.e., deviations from the free string on the permutation orbifold become arbitrarily small at large t_0. Just as in the previous section, supersymmetry restoration at late times becomes a simple corollary of the above bound.

4 Adiabaticity

In the preceding sections we have derived bounds controlling the convergence towards the late-time limit of time-dependent matrix theories. A reader familiar with quantum adiabatic theory (see, e.g., [26]) will immediately recognize the structure of our manipulations. Indeed, for both 11-dimensional and 10-dimensional cases, we have related the time-dependent theory to properties of time-independent theories (flat space matrix theory, super-Yang-Mills theories), which is characteristic of the adiabatic approximation. In this section, we shall briefly review quantum adiabatic theory and display its connections to the derivations in the previous sections.

Given a general quantum system with a time-dependent Hamiltonian $H(t)$ and a state vector $|\Psi\rangle$ satisfying the Schrödinger equation

$$i\frac{d}{dt}\,|\Psi\rangle = H(t)\,|\Psi\rangle\,, \tag{4.1}$$

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one can always expand the state vector in a (time-dependent) basis of *instantaneous* eigenvectors $|\Psi_n(t)\rangle$ of $H(t)$:

$$|\Psi\rangle = \sum_n c_n(t)\, |\Psi_n(t)\rangle\,, \qquad H(t)\, |\Psi_n(t)\rangle = E_n(t)\, |\Psi_n(t)\rangle\,. \tag{4.2}$$

Note that our summation over n is a symbolic notation that implies integration over the continuous spectrum and summation over all discrete eigenvector labels, and none of our derivations use any assumptions about discreteness of the spectrum. The Schrödinger equation then takes the form

$$i\frac{dc_n}{dt} + i\sum_m c_m(t)\, \langle\Psi_n(t)|\, \frac{d}{dt}\, |\Psi_m(t)\rangle = E_n(t)c_n(t). \tag{4.3}$$

An adiabatic regime occurs when the second term on the left hand side becomes small. Heuristically, this happens when $H(t)$ varies slowly in some sense, and so do $|\Psi_n(t)\rangle$, so that their derivatives can be neglected. In general, it is difficult to spell out more handy conditions for the emergence of this regime. However, in particular cases, simple and explicit adiabatic parameters can be constructed, as we shall see below.

If the second term on the left hand side of (4.3) can indeed be neglected, the equations can be solved trivially to yield

$$c_n(t) = C_n \exp\left[-i\int dt E_n(t)\right], \qquad |\Psi\rangle = \sum_n C_n \exp\left[-i\int dt E_n(t)\right] |\Psi_n(t)\rangle\,. \tag{4.4}$$

Note the close similarity between these approximate adiabatic solutions to the time-dependent Schrödinger equation and the familiar solutions to a time-independent Schrödinger equation: $|\Psi\rangle = \sum_n C_n \exp\left[-iE_n t\right] |\Psi_n\rangle$. (The stationary eigenvectors $|\Psi_n\rangle$ are simply replaced by the instantaneous eigenvectors $|\Psi_n(t)\rangle$, and the $E_n t$ in the phase factors are simply replaced by $\int dt E_n(t)$.)

The relation between the time-dependent and time-independent systems becomes even more straightforward if the instantaneous spectrum $E_n(t)$ of $H(t)$ scales uniformly as a function of time, namely,

$$E_n(t) = \lambda(t) E_n^{(0)}, \tag{4.5}$$

where the $E_n^{(0)}$ do not depend on time. In that case, the phase factors in (4.4) become simply $\left[-iE_n^{(0)}\int dt\lambda(t)\right]$, in other words, they differ from the stationary case only by replacing t with $\int dt\lambda(t)$, *independently* of which state vector one is dealing with.

For the class of systems characterized by (4.5), it is convenient to perform a variable redefinition in the Schrödinger equation. Eq. (4.5) implies that there exists a time-dependent unitary transformation $S(t)$ such that

$$S^\dagger(t)H(t)S(t) = \lambda(t)H_0, \tag{4.6}$$

where H_0 does not depend on time and possesses the spectrum $E_n^{(0)}$. One can then introduce $|\Phi\rangle = S^\dagger(t)\, |\Psi\rangle$, satisfying

$$i\frac{d}{dt}\, |\Phi\rangle + i\left(S^\dagger(t)\frac{d}{dt}S(t)\right) |\Phi\rangle = \lambda(t)H_0\, |\Phi\rangle\,. \tag{4.7}$$

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An adiabatic regime occurs when the second term on the left hand side can be neglected compared to the right hand side. The time dependence responsible for this relation between the two terms can be isolated into the operator

$$\frac{1}{\lambda(t)}\left(S^\dagger(t)\frac{d}{dt}S(t)\right). \tag{4.8}$$

If the second term on the left hand side of (4.7) can indeed be neglected, the equation is solved to yield

$$|\Psi(t)\rangle = S(t)\,|\Phi(t)\rangle = S(t)\exp\left[-iH_0\int_{t_0}^{t} dt\lambda(t)\right]S^\dagger(t_0)\,|\Psi(t_0)\rangle \tag{4.9}$$

The evolution is essentially that of a stationary system described by H_0, except that a unitary transformation $S(t)$ is performed and the "time flow" is deformed from t to $\int dt\lambda(t)$.

The situation simplifies further if $S(t)$ corresponds to a (time-dependent) linear transformation of the canonical coordinates

$$q_k = \mathbf{s}_{kl}(t)\tilde{q}_l \tag{4.10}$$

(and the corresponding transformation of the canonical momenta):

$$S(t) = \sqrt{\det\mathbf{s}}\int dq\,|\mathbf{s}\,q\rangle\,\langle q| = \frac{1}{\sqrt{\det\mathbf{s}}}\int dq\,|q\rangle\,\langle\mathbf{s}^{-1}q|\,. \tag{4.11}$$

Differentiating $S(t)$ with respect to t is somewhat subtle and is most conveniently performed using

$$\begin{aligned}\frac{d}{dt}\left|\mathbf{s}^{-1}q\right\rangle &\equiv \frac{d}{dt}\delta(x-\mathbf{s}^{-1}q) = (-\mathbf{s}^{-1}q)^{\bullet}_k\,\partial_k\delta(x-\mathbf{s}^{-1}q)\\ &= i(\mathbf{s}^{-1}\dot{\mathbf{s}}\mathbf{s}^{-1}q)_k\hat{p}_k\delta(x-\mathbf{s}^{-1}q) \equiv i(\mathbf{s}^{-1}\dot{\mathbf{s}})_{kl}\hat{p}_k\hat{q}_l\left|\mathbf{s}^{-1}q\right\rangle.\end{aligned} \tag{4.12}$$

Then,

$$\frac{d}{dt}S = S\left(-\frac{(\det\mathbf{s})^{\bullet}}{2\det\mathbf{s}} - i(\mathbf{s}^{-1}\dot{\mathbf{s}})_{kl}q_l p_k\right) = S\left(-\frac{i}{2}(\mathbf{s}^{-1}\dot{\mathbf{s}})_{kl}\{q_l,p_k\}\right). \tag{4.13}$$

Hence, (4.7) takes the form

$$i\frac{d}{dt}\,|\Phi\rangle + \frac{1}{2}(\mathbf{s}^{-1}\dot{\mathbf{s}})_{kl}\{q_l,p_k\}\,|\Phi\rangle = \lambda(t)H_0\,|\Phi\rangle\,. \tag{4.14}$$

All the time dependences have now been isolated into numerical pre-factors of the operators. The characteristic ratios (adiabatic parameters) quantifying the neglect of the second term on the left hand side with respect to the right hand side can now be given[4] as a matrix:

$$\frac{\mathbf{s}^{-1}\dot{\mathbf{s}}}{\lambda(t)}\quad. \tag{4.15}$$

[4]For the simple systems we are now considering, a purely classical consideration involving the transformation $q_k = \mathbf{s}_{kl}(t)q'_l$ would produce the same adiabatic parameters. The way we have derived them here implies automatically their validity for the quantum case, which is the physically relevant regime for gravitational matrix models.

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One can examine how the above formulas work for the familiar case of a time-dependent harmonic oscillator with $H(t) = (p^2 + \omega^2(t)x^2)/2$. If one makes use of (4.11) corresponding to the transformation $x = \tilde{x}/\sqrt{\omega}$, one obtains $S^\dagger(t)H(t)S(t) = \omega(p^2 + x^2)/2$. In other words, our above analysis applies with $\mathbf{s} = 1/\sqrt{\omega}$ and $\lambda = \omega$, so that (4.15) becomes simply proportional to $\dot{\omega}/\omega^2$, i.e., the relative change of ω per period of the oscillations, which is the familiar adiabatic parameter of the harmonic oscillator. The new Hamiltonian after the transformation has been effected can be read off (4.14) as

$$\tilde{H} = \frac{\omega}{2}(p^2 + x^2) + \frac{\dot{\omega}}{4\omega}(px + xp). \tag{4.16}$$

We now turn to the derivations of the preceding sections. The variable redefinition (2.4) we have employed for the 11-dimensional matrix theory is precisely of the form (4.10), and it converts the time-dependent system to a time-independent system plus a correction decaying at large times. If we forget about the geometrical interpretation of this variable redefinition, it simply becomes a particular case of (4.10) demonstrating that the (superficially steep) time dependences in the Rosen form matrix theory action (2.3) in fact become adiabatic at late times, and the system is well approximated by its time-independent counterpart, i.e., the flat space matrix theory.

For the 10-dimensional case, equations (3.17)–(3.18) are precisely of the form (4.2)–(4.3). Furthermore, (4.5) is almost satisfied at late times (because the spectrum approaches a constant limit). Our analysis of the 10-dimensional case shows explicitly that the second term on the left-hand side of (3.18) can be neglected in comparison to the other terms (provided that the convergence of time-independent matrix string theories to the Dijkgraaf-Verlinde-Verlinde limit is sufficiently tame), which is by definition the adiabatic regime.

One may try to object that adiabaticity is not a relevant term for our discussions, since a variable redefinition brings the equations of motion to the form where all the time dependent terms become small at large times. This objection is vacuous, however, since it would also apply to a wide range of systems commonly thought of as adiabatic. Indeed, if $E_n(t)$ in (4.3) approaches a constant limit at late times, (4.3) will take the form where all the time dependent terms become small at large times whenever an adiabatic regime occurs at late times, in this general setting. Likewise, the familiar time-dependent harmonic oscillator $H(t) = (p^2 + \omega^2(t)x^2)/2$, the simplest system used for text book demonstrations on adiabaticity, can be converted to

$$H(t) = \frac{p^2 + x^2}{2} + \frac{\dot{\omega}}{4\omega^2}(px + xp), \tag{4.17}$$

if we start from (4.16) and introduce a new time variable $\tau = \int dt\, \omega(t)$ (the dot in the above formula still denotes the derivative with respect to the old time t to maintain the familiar expression for the adiabatic parameter, $\dot{\omega}/\omega^2 = \partial_\tau \omega/\omega$). Then there is a one-to-one correspondence between the adiabatic regime and the (new) Hamiltonian being almost constant: both occur when the adiabatic parameter $\dot{\omega}/\omega^2$ is small. Yet, this mathematical structure does not prevent anyone from employing the term 'adiabatic' for the adiabatic regime of a time-dependent harmonic oscillator.

There appears to be widespread intuition that adiabaticity in quantum mechanics is somehow connected to discreteness of the energy spectrum, and adiabatic parameters emerge from comparisons of the rate of change of various terms in the Hamiltonian to energy spacings in the discrete spectrum. This intuition stems from the simplest versions of adiabatic theorems (proved, for example, in [26]), which are sufficient, but certainly not necessary conditions for adiabaticity.[5] To dispel the doubts regarding adiabaticity in systems with a continuous spectrum, one may simply notice that an inverted harmonic oscillator $H(t) = (p^2 - \omega^2(t)x^2)/2$ can be brought to the form

$$H(t) = \frac{p^2 - x^2}{2} + \frac{\dot{\omega}}{4\omega^2}(px + xp) \tag{4.18}$$

by the same transformations we used in obtaining (4.17). This system will be adiabatic whenever $\dot{\omega}/\omega^2$ is small by virtue of the bound (2.12), irrespectively of the fact that the spectrum is entirely continuous. For the matrix theories we have explored in this article, adiabaticity has been proved by constructing explicit bounds on deviations from the strictly adiabatic evolution.

5 Conclusions

We have considered the low-curvature regime of time-dependent matrix theories and matrix string theories, and displayed the relation between the emergence of near-classical space-time and adiabaticity of the time dependences in the matrix theory actions. In this context, supersymmetry of the matrix theories (explicitly broken by the time dependence) is naturally restored at low curvatures, and the conventional space-time interpretation of matrix theories becomes viable.

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[5] One must also keep in mind that there are different ways to reach the limit of slow relative variation of the Hamiltonian, and they may result in different adiabatic parameters. For example, in adiabatic theorems proved in [26], one considers a *general* Hamiltonian $H(sT)$ where s changes between 0 and 1, and the limit of slow variation is reached by sending T to infinity. The evolution is examined between $sT = 0$ and $sT = T$, and demanding adiabaticity is a very strong requirement, since the non-adiabatic terms should produce a negligible contribution even over the huge time interval T. We study adiabatic evolution on finite time intervals for Hamiltonians of a very particular form.

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Thermalization of Strongly Coupled Field Theories

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Using the holographic mapping to a gravity dual, we calculate 2-point functions, Wilson loops, and entanglement entropy in strongly coupled field theories in $d = 2$, 3, and 4 to probe the scale dependence of thermalization following a sudden injection of energy. For homogeneous initial conditions, the entanglement entropy thermalizes slowest and sets a time scale for equilibration that saturates a causality bound. The growth rate of entanglement entropy density is nearly volume-independent for small volumes but slows for larger volumes. In this setting, the UV thermalizes first.

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It is widely believed that the observed nearly inviscid hydrodynamics of relativistic heavy ion collisions at collider energies is an indication that the matter produced in these nuclear reactions is strongly coupled [1]. Some such strongly coupled field theories can be studied by using the holographic duality between gravitational theories in asymptotically anti–de Sitter (AdS) space-times and quantum field theories on the boundary of AdS. The thermal state of the field theory is represented by a black brane in AdS, and near-equilibrium dynamics is studied in terms of perturbations of the black hole metric. A key remaining challenge is to understand the far from equilibrium process of thermalization. The AdS/CFT correspondence relates the approach to thermal equilibrium in the boundary theory to black hole formation in the bulk.

Recent works studied the gravitational collapse of energy injected into AdS_5 and the formation of an event horizon [2]. These works started from locally anisotropic metric perturbations near the AdS boundary and studied the rate at which isotropic pressure was established by examining the evolution of the stress tensor. By studying gravitational collapse induced by a small scalar perturbation, the authors of Ref. [3] concluded that local observables behaved as if the system thermalized almost instantaneously. Here we model the equilibrating field configuration in AdS by an infalling homogeneous thin mass shell [4,5] and study how the rate of thermalization varies with spatial scale and dimension. We consider $2d$, $3d$, and $4d$ field theories dual to gravity in asymptotically AdS_3, AdS_4, and AdS_5 space-times, respectively. Our treatment of $2d$ field theories is analytic.

Expectation values of local gauge-invariant operators, including the energy-momentum tensor and its derivatives, provide valuable information about the applicability of viscous hydrodynamics but cannot be used to explore the scale dependence of deviations from thermal equilibrium. Equivalently, in the dual gravitational description these quantities are sensitive only to the metric close to the AdS boundary. Nonlocal operators, such as Wilson loops and 2-point correlators of gauge-invariant operators, probe the thermal nature of the quantum state on extended spatial scales. In the AdS language, these probes reach deeper into the bulk space-time, which corresponds to probing further into the infrared of the field theory. They are also relevant to the physics probed in relativistic heavy ion collisions, e.g., through the jet quenching parameter $\hat{q}$ [6] and the color screening length.

A global probe of thermalization is the entanglement entropy S_A [7,8] of a domain A, measured after subtraction of its vacuum value. In the strong coupling limit, it has been proposed that S_A for a region A with boundary ∂A in the field theory is proportional to the area of the minimal surface γ in AdS whose boundary coincides with ∂A: $S_A = \mathrm{Area}(\gamma)/4G_N$, where G_N is Newton's constant [8]. Thus, for a ($d = 2$)-dimensional field theory, S_A is the length of a geodesic curve in AdS_3 (studied in Ref. [9]); for $d = 3$, S_A is the area of a $2d$ sheet in AdS_4 (studied in Ref. [10]); and for $d = 4$, S_A is the volume of a $3d$ region in AdS_5. In $d = 3$ the exponential of the area of the minimal surface that measures S_A also computes the expectation value of the Wilson loop that bounds the minimal surface. Wilson loops in $d = 4$ correspond to $2d$ minimal surfaces as well.

First, we consider equal-time 2-point correlators of gauge-invariant operators $\mathcal{O}$ of large conformal dimension Δ. In the dual supergravity theory this correlator can be expressed, in the semiclassical limit, in terms of the length

$\mathcal{L}(\mathbf{x}, t)$ of the bulk geodesic curve that connects the end points on the boundary: $\langle \mathcal{O}(\mathbf{x}, t)\mathcal{O}(0, t)\rangle \sim \exp[-\Delta \mathcal{L}(\mathbf{x}, t)]$ [11]. When multiple such geodesics exist, one has to consider steepest descent contours to determine the contribution from each geodesic.

We consider a $(d+1)$-dimensional infalling shell geometry described in Poincaré coordinates by the Vaidya metric

$$ds^2 = \frac{1}{z^2}\{-[1 - m(v)z^d]dv^2 - 2dzdv + d\mathbf{x}^2\}, \qquad (1)$$

where v labels ingoing null trajectories, and we set the AdS radius to 1. The boundary is at $z = 0$, where v coincides with the observer time t. The mass function of the infalling shell is

$$m(v) = (M/2)[1 + \tanh(v/v_0)], \qquad (2)$$

where v_0 determines the thickness of a shell falling along $v = 0$. The metric interpolates between vacuum AdS inside the shell and an AdS black brane geometry with Hawking temperature $T = dM^{1/d}/4\pi$ outside the shell. 2-point functions agree with those of a boundary field theory at thermal equilibrium only if they are dominated by geodesics that stay outside the shell.

The geodesic length $\mathcal{L}$ diverges due to contributions near the AdS boundary. We introduce an ultraviolet cutoff z_0 and define a renormalized correlator $\delta\mathcal{L} = \mathcal{L} + 2\ln(z_0/2)$ by removing the divergent part of the correlator in the vacuum state (pure AdS). The renormalized equal-time 2-point function is $\langle \mathcal{O}(\mathbf{x}, t)\mathcal{O}(0, t)\rangle_{\rm ren} \sim \exp[-\Delta\delta\mathcal{L}(\mathbf{x}, t)]$. We compute the renormalized correlator as a function of $\mathbf{x}$ and t in a state evolving towards thermal equilibrium and compare it to the corresponding thermal correlator. In the bulk, this amounts to computing geodesic lengths in a collapsing shell geometry and comparing them to geodesic lengths in the black brane geometry ($\delta\mathcal{L}_{\rm thermal}$) resulting from the collapse.

We study geodesics with boundary separation ℓ in the x direction in AdS_3, AdS_4, and AdS_5 modified by the infalling shell. The end point locations are denoted as $(v, z, x) = (t_0, z_0, \pm\ell/2)$, where z_0 is the UV cutoff. The lowest point of the geodesic in the bulk is the midpoint located at $(v, z, x) = (v_*, z_*, 0)$. Geodesics are obtained by solving differential equations for the functions $v(x)$ and $z(x)$ with these boundary conditions and are unique in the infalling shell background. The length of the geodesics is $\mathcal{L}(\ell, t_0) = 2\int_0^{\ell/2} dx z_* z(x)^{-2}$. In empty AdS, this gives the renormalized geodesic length $\delta\mathcal{L}_{\rm AdS} = 2\ln(\ell/2)$.

A numerical solution for the length of geodesics crossing the shell in the $d = 2$ (AdS_3) case was obtained in Ref. [9]. We checked that physical results do not depend significantly on the shell thickness when v_0 is small and then derived an analytical solution in the $v_0 \to 0$ limit:

$$\delta\mathcal{L}(\ell, t_0) = 2\ln\left[\frac{\sinh(\sqrt{M}t_0)}{\sqrt{M}s(\ell, t_0)}\right], \qquad (3)$$

where $s(\ell, t_0) \in [0, 1]$ is parametrically defined by

$$\ell = \frac{1}{\sqrt{M}}\left[\frac{2c}{s\rho} + \ln\left(\frac{2(1+c)\rho^2 + 2s\rho - c}{2(1+c)\rho^2 - 2s\rho - c}\right)\right],$$
$$2\rho = \coth(\sqrt{M}t_0) + \sqrt{\coth^2(\sqrt{M}t_0) - \frac{2c}{c+1}}, \qquad (4)$$

with $c = \sqrt{1 - s^2}$ and $\rho = (\sqrt{M}z_c)^{-1}$. Here z_c is the radial location of the intersection between the geodesic and the shell. For any given ℓ, at sufficiently late times, the geodesic lies entirely in the black brane background outside the shell. In this case the length is

$$\delta\mathcal{L}_{\rm thermal}(\ell) = 2\ln[(1/\sqrt{M})\sinh(\sqrt{M}\ell/2)], \qquad (5)$$

representing the result for thermal equilibrium.

We use these analytic relations in $d = 2$ and find $\delta\mathcal{L}(\ell, t_0)$ in $d = 3, 4$ by numerical integration. We measure the approach to thermal equilibrium by comparing $\delta\mathcal{L}$ at any given time with the late time thermal result (see Fig. 1). In any dimension, this compares the logarithm of the 2-point correlator at different spatial scales with the logarithm of the thermal correlator. For $d = 2$, the same quantity measures by how much the entanglement entropy at a given spatial scale differs from the entropy at thermal equilibrium.

Various thermalization times can be extracted from Fig. 1. For any spatial scale we can ask for (a) the time $\tau_{\rm dur}$ until full thermalization (measured as the time when the geodesic between two boundary points just grazes the infalling shell), (b) the half-thermalization time $\tau_{1/2}$, which measures the duration for the curves to reach half of their equilibrium value, and (c) the time $\tau_{\rm max}$ at which thermalization proceeds most rapidly, namely, the time for which the curves in Fig. 1 are steepest. These are plotted in Fig. 2. In $d = 2$ we can analytically derive the linear relation $\tau_{\rm dur} \equiv \ell/2$, as also observed in Ref. [9].

The linearity of $\tau_{\rm dur}(\ell)$ in $2d$ is expected from general arguments in conformal field theory [7], and the coefficient is as small as possible under the constraints of causality. The thermalization time scales $\tau_{1/2}$ and $\tau_{\rm max}$ for $3d$ and $4d$ field theories (Fig. 2, middle and right) are sublinear in the

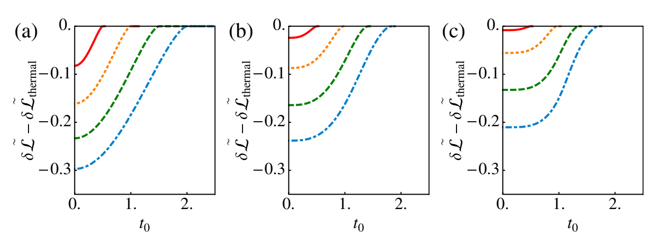

FIG. 1 (color online). $\delta\tilde{\mathcal{L}} - \delta\tilde{\mathcal{L}}_{\rm thermal}$ ($\tilde{\mathcal{L}} \equiv \mathcal{L}/\ell$) as a function of boundary time t_0 for $d = 2, 3, 4$ (left, right, middle) for a thin shell ($v_0 = 0.01$). The boundary separations are $\ell = 1, 2, 3, 4$ (top to bottom curve). All quantities are given in units of M. These numerical results match analytical results for $d = 2$ as $v_0 \to 0$.

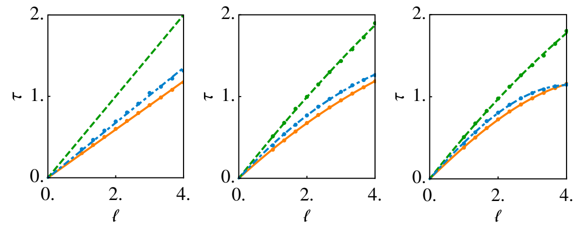

FIG. 2 (color online). Thermalization times ($\tau_{\rm dur}$, top line; $\tau_{\rm max}$, middle line; $\tau_{1/2}$, bottom line) as a function of spatial scale for $d = 2$ (left), $d = 3$ (middle), and $d = 4$ (right) for a thin shell ($v_0 = 0.01$). All thermalization time scales are linear in ℓ for $d = 2$ and deviate from linearity for $d = 3, 4$.

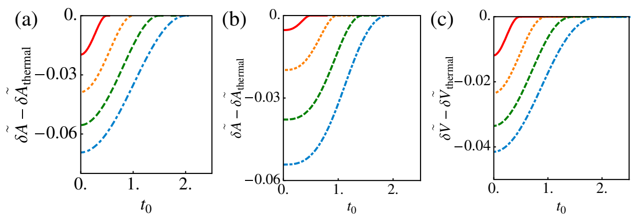

FIG. 3 (color online). $\delta\tilde{\mathcal{A}} - \delta\tilde{\mathcal{A}}_{\rm thermal}$ ($\tilde{\mathcal{A}} \equiv \mathcal{A}/\pi R^2$; left and middle panels) and $\delta\tilde{V} - \delta\tilde{V}_{\rm thermal}$ [$\tilde{V} \equiv V/(4\pi R^3/3)$; right panel] as a function of t_0 for radii $R = 0.5, 1, 1.5, 2$ (top curve to bottom curve) and mass shell parameters $v_0 = 0.01$, $M = 1$, in $d = 3$ (left panel) and $d = 4$ (middle and right panel) field theories.

spatial scale. In the range we study, the complete thermalization time $\tau_{\rm dur}$ deviates slightly from linearity and is somewhat shorter than $\ell/2$. We will later discuss whether a rigorous causality bound for thermalization processes exists or not.

In $2d$ "quantum quenches" where a pure state prepared as the ground state of a Hamiltonian with a mass gap is followed as it evolves according to a different, critical Hamiltonian, a nonanalytic feature was found where thermalization at a spatial scale ℓ is completed abruptly at $\tau_{\rm dur}(\ell)$ [7,9]. An analogous feature is evident in Fig. 1 (left) as a sudden change in the slope at $\tau_{\rm dur}$, smoothed out only by the small nonzero thickness of the shell or, equivalently, by the intrinsic duration of the injection of energy. We find a similar (higher-order) nonanalyticity for $d = 3, 4$ (Fig. 1, middle and right) and expect this to be a general consequence of abrupt injection of energy in any dimension.

Figure 2 shows that complete thermalization of the equal-time correlator is first observed at short length scales or large momentum scales (see also [5]). While this behavior follows directly in our setup with a shell falling in from the ("UV") boundary of AdS, this "top-down" thermalization contrasts with the behavior of weakly coupled gauge theories even with energy injected in the UV. In the "bottom-up" scenario [12] applicable to that case, hard quanta of the gauge field do not equilibrate directly by randomizing their momenta but gradually degrade their energy by radiating soft quanta, which fill up the thermal phase space and equilibrate by collisions among themselves. This bottom-up scenario is linked to the infrared divergence of the splitting functions of gauge bosons and fermions in perturbative gauge theory. It contrasts with the "democratic" splitting properties of excitations in strongly coupled super Yang-Mills theory that favor an approximately equal sharing of energy and momentum [13].

The thermal limit of the Wightman function that we studied above is a necessary but not a sufficient condition for complete thermalization. To examine whether thermalization proceeds similarly for other probes, we also studied entanglement entropy and spacelike Wilson loop expectation values in $3d$ (following [10]) and $4d$ field theories. Entanglement entropy in $3d$ field theories is holographically related to minimal surfaces in AdS_4 and hence to the logarithm of the expectation value of Wilson loops. We considered circular loops of radius R in $d = 3, 4$. The minimal spacelike surface in AdS_{d+1} whose boundary is this circular loop extends into the bulk space radially and into the past. The tip occurs at $(v_*, z_*, \mathbf{x} = \mathbf{0})$. The cross section at fixed z and v is a circle, and thus the surface is parameterized in terms of the radii ρ of these circles. The overall shape minimizes the action for the two functions $z(\rho)$ and $v(\rho)$:

$$\mathcal{A}[R] = 2\pi \int_0^R d\rho \frac{\rho}{z^2}\sqrt{1 - [1 - m(v)z^d]v'^2 - 2z'v'}, \quad (6)$$

where $z'(\rho) = dz/d\rho$, etc. The resulting Euler-Lagrange equations can be numerically integrated. We regularize the area by subtracting the divergent piece of the area in "empty" AdS: $\delta\mathcal{A}[R] = \mathcal{A}[R] - (R/z_0)$. Entanglement entropy of spherical volumes in $d = 4$ is similarly computed in terms of minimal volumes in AdS_5 by minimizing an equation similar to (6) and defining $\delta V[R]$ by subtracting the divergent volume in empty AdS.

The deficit area $\delta\mathcal{A} - \delta\mathcal{A}_{\rm thermal}$ for Wilson loops in $d = 3, 4$ and the deficit volume $\delta V - \delta V_{\rm thermal}$ are plotted in Fig. 3 for several boundary radii R as a function of the boundary time t_0. By subtracting the thermal values, we can observe the deviation from equilibrium for each spatial scale at a time t_0. Comparing the three thermalization times defined earlier as a function of the loop diameter (Fig. 4), we find that for the entanglement entropy in $d = 3, 4$, the complete thermalization time $\tau_{\rm dur}(R)$ is close to being a straight line with unit slope over the range of scales that we

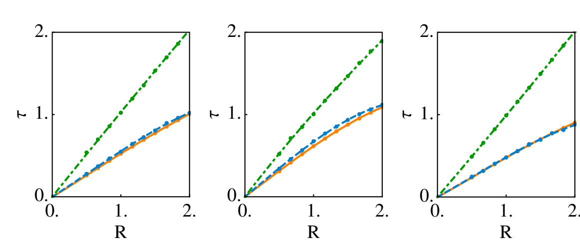

FIG. 4 (color online). Thermalization times ($\tau_{\rm dur}$, top line; $\tau_{\rm max}$, middle line; $\tau_{1/2}$, bottom line) as a function of the diameter for circular Wilson loops in $d = 3, 4$ (left, middle) and for entanglement entropy of spherical regions in $d = 4$ (right).

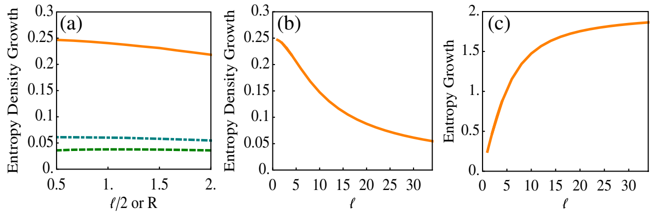

FIG. 5 (color online). (Left) Maximal growth rate of entanglement entropy *density* vs radius of entangled region for $d = 2, 3, 4$ (top to bottom). (Middle) The same plot for $d = 2$, larger range of ℓ. (Right) Maximal entropy growth rate for $d = 2$.

study (as observed in [10] for $d = 3$). On the other hand, for Wilson loops in $d = 4$, $\tau_{\rm dur}(R)$ deviates somewhat from linearity and is shorter than R.

Our thermalization times for Wilson loop averages and entanglement entropy seem remarkably similar to those for 2-point correlators (after noting that R here is the radius of the thermalizing region and ℓ in Fig. 2 is the diameter). Slightly "faster-than-causal" thermalization, possibly due to the homogeneity of the initial configuration, seems to occur for the probes that do not correspond to entanglement entropy in each dimension. For the latter, the thermalization time is linear in the spatial scale and saturates the causality bound. As the actual thermalization rate of a system is set by the slowest observable, our results suggest that in strongly coupled theories with a gravity dual, thermalization occurs "as fast as possible" at each scale, subject to the constraint of causality. Taking the thermal scale $\ell \sim \hbar/T$ as the length scale, this suggests that for strongly coupled matter $\tau_{\rm dur} \sim 0.5\hbar/T$, in particular, $\tau_{\rm dur} \sim 0.3$ fm$/c$ at heavy ion collider energies ($T \approx 300$–400 MeV), comfortably short enough to account for the experimental observations.

The average growth rate of the *coarse grained* entropy in nonlinear dynamical systems is measured by the Kolmogorov-Sinaï (KS) entropy rate $h_{\rm KS}$ [14], which is given by the sum of all positive Lyapunov exponents. For a classical SU(2) lattice gauge theory in $4d$, $h_{\rm KS}$ has been shown to be proportional to the volume [15]. For a system starting far from equilibrium, the KS entropy rate generally describes the rate of growth of the coarse grained entropy during a period of linear growth after an initial dephasing period and before the close approach to equilibrium [16]. Here we observe similar linear growth of *entanglement* entropy density in $d = 2, 3, 4$ [Figs. 1(a), 3(a), and 3(c)]. For small boundary volumes, the growth rate of entropy *density* is nearly independent of the boundary volume [almost parallel slopes in Figs. 1(a), 3(a), and 3(c) and nearly constant maximal growth rate in Fig. 5(a)]. Equivalently, the growth rate of the entropy is proportional to the volume—suggesting that entropy growth is a local phenomenon. However, in $d = 2$ where our analytic results enable study of large boundary volumes ℓ, we find that the growth rate of the entanglement entropy density changes for large ℓ, falling asymptotically as $1/\ell$ [Fig. 5(b)]. Equivalently, the entropy has a growth rate that approaches a constant limiting value for large ℓ [Fig. 5(c)] and thus cannot arise from a local phenomenon. This behavior suggests that entanglement entropy and coarse grained entropy have different dynamical properties.

We have investigated the scale dependence of thermalization following a sudden injection of energy in $2d$, $3d$, and $4d$ strongly coupled field theories with gravity duals. The entanglement entropy sets a time scale for equilibration that saturates a causality bound. The relationship between the entanglement entropy growth rate and the KS entropy growth rate defined by coarse graining of the phase space distribution raises interesting questions.

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